Where There is No Teacher

Basic Applied Mathematics

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Contents

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Preface

There is a great need for A-Level textbooks in Tanzania that have been written to exactly address the syllabi. The whole country is plagued by a lack of teachers, but especially A-level science and mathematics teachers are hard to come by. It's horrible to deprive the country's best and brightest students of a good chance at education. I think that an inexpensive Basic Applied Mathematics textbook could greatly help A-Level science students, especially at those schools where there is no BAM teacher.

This book is written specifically to meet the needs of the BAM student:

- It should be inexpensive.
- It goes straight to the point. For better or for worse, BAM students want to learn enough mathematics to do problems in their subjects, currently and should the continue studying, so that they can spend more time studying their combination subjects.
- It should address especially those topics which frequently appear on NECTA exams.
- In addition to the A-Level syllabus topics, it should provide a quick review of important O-Level topics.
- It should be light on the theory behind the math, but have many worked examples and exercises, included many problems from past NECTA exams.
- Whenever possible, it should address HIV/AIDS related issues.

To The Student

BAM is an important subject. It's not one of your combination subjects, but it will help you with them. This book was written so that you can learn even if you don't have a teacher. Of course it's better if you do have a teacher to guide you through the subject. But even if you are in a place where there is no teacher, you can still learn lots by reading this book and doing the exercises.

I have included many exercises, both for practice and from past NECTA papers. In the back, you will find solutions to some problems, and hints to others. Please, try your level best *before* looking for the answer, because on the NECTA exam there are no answers in the back.

In Appendix D, you can find instructions on how to use a condom. This is not related to mathematics, but all the education in the world will not help you live through AIDS. The best way to avoid AIDS is abstinence. However, if you do have sex, you can still protect yourself by using a condom. Using a condom protects you, protects your partner, protects your future husband or wife, and your future children. Using a condom some of the time is only a little better than never using a condom. *To completely protect yourself, you must use a condom* ***every*** *time you have sex.*

Some advice for all your NECTA exams:

- When it's time to take the exam, you have been studying for 2 years. The night before the exam, do not study late at night, 1 more hour of sleep will help you more than 1 more hour of studying.
- In the morning just before the exam, eat something! Not a full meal to make you tired, but something to give you some energy. If you are hungry, you will not be able to think well.
- Also try to get a little exercise right before the exam, like jogging for 10 minutes. It will help to wake up your body and your mind.
- When you sit down to take the exam, first read the entire test and choose which problems you want to do. Try not to waste time on problems you don't know well until you have already finished the ones you can do easily.

Good Luck!

Chapter 1

The Basics

Every student of Basic Applied Mathematics should know the contents of this chapter thoroughly. Most of the topics covered are in the O-Level syllabus, they have been selected as the most important topics from O-Level that form the foundation that we will build upon in covering the BAM syllabus.

Of course, in your O-Level studies you may not have thoroughly covered all of these topics, but the time to learn is now. This knowledge is necessary to succeed in A-Level, and if you are uncomfortable or slow in performing calculations from O-Level, you will find yourself out of time on your A-Level NECTA exam. Also, some O-Level topics like functions and their inverses (Form III) appear again in the A-Level syllabus, so they are covered here.

So, go through this chapter quickly, but if you find a topic or a method you are not familiar with, get adequate practice. Especially look at the factoring and unit circle sections. If you are an expert in these areas, you will find yourself able to complete problems much faster.

1.1 Algebra Revision: E x p a n d i n g and F(actoring)

1.1.1 Common Mistakes

First, we must address a common mistake that the A-Level student should never make:

$$(a+b)^2 = (a+b)\cdot(a+b) = a^2 + 2ab + b^2 \tag{1.1}$$

We will not write here the mistake that is made. So many people forget about the $2ab$ term, you should be smarter than them! Similarly,

$$(a-b)^2 = (a-b)\cdot(a-b) = a^2 - 2ab + b^2, \tag{1.2}$$

However,

$$a^2 - b^2 = (a+b)(a-b) \tag{1.3}$$

Look at the difference between Equations 2 and 3. They are different! Know both of these, but don't confuse them! For more detail on expanding $(a+b)^n$ see section 6.4.

1.1.2 The X Factor: Factoring Polynomials

'Factor' means to show how different parts multiply to make the product. In factoring an expression, you want as many different things times each other as possible. You put in parentheses ( ). 'Expand' is the opposite. After you expand, there should be no parentheses at all. We need to address some issues with polynomials. They are covered in more detail in section 6.1; this is just a foundation.

We usually care about what is x when a polynomial is 0. Factoring is a very fast way to find these values of x. For any numbers, we know that if $a \times b = 0$, then either a is 0 or b is 0, or even both. Factoring takes this one step farther. If $(x+a)(x+b)=0$, then $(x+a)$ is 0 or $(x+b)$ is 0, which means that $x=-a$ or $x=-b$.

If we expand $(x+a)(x+b)$ we get $(x+a)(x+b)=x^2+ax+bx+ab=x^2+(a+b)x+ab$. Factoring is doing this backwards. For example, to factor x^2+3x+2 we want to make it look like $(x+a)(x+b)$. If we line it up,

$$\begin{array}{ccccc} x^2 & + & 3 & x & + & 2 \\ x^2 & + & (a+b) & x & + & ab \end{array}$$

so we need to think of numbers a and b such that $(a+b)=3$ and $ab=2$. Can you think of any? The product $a\cdot b$ is a good place to start looking. What numbers multiply to make 2? There are two possibilities, $2=1\times 2$ and $2=-1\times -2$. Now, what are the sums? $-1+-2=-3$, that is not what we want, but $1+2=3$, so it fits! Therefore

$$x^2+3x+2=(x+1)(x+2)$$

This helps us find the roots, or the values of x to make the polynomial 0 because if

$$x^2+3x+2=(x+1)(x+2)=0$$

then, as we said before either $x+1=0$, which means $x=-1$; or $x+2=0$, which means $x=-2$. So the roots of this polynomial are $x=-1$ or -2.

Ex 1: *Factor* x^2+4x-5.

Solution: Think of numbers a and b such that $a+b=4$ and $ab=-5$. Starting with the product, possibilities are 1 and -5 or -1 and 5. The sum tells us that the answer is -1 and 5. Thus

$$x^2+4x-5=(x+5)(x-1)$$

is our factored polynomial. ✠

Ex 2: *Factor* $2x^2-12x+16$.

Solution: First we divide factor out a 2, to get $2(x^2-6x+8)$. Then, think of numbers a and b such that $a+b=-6$ and $ab=8$. Because the sum is negative, we know at least one of the numbers is negative. But the product is positive so they must both be negative or both be positive. Thus they are both negative. Possibilities for the product are -1 and -8, or -2 and -4. The sum tells us that the answer is -2 and -4. Thus

$$2x^2-12x+16=2(x-2)(x-4)$$

is our factored polynomial. ✠

Ex 3: *Factor* $3x^2-x-2$.

Solution: Sometimes, like here, we cannot divide by 2, but because there is a coefficient of x^2 we can start with

$$(3x+a)(x+b)$$

But we know that $ab=-2$, and that $a+3b=-1$. A little thought, and we find that $a=2$, $b=-1$. Just try a few numbers and you can usually get it. Thus

$$3x^2-x-2=(3x+2)(x-1).$$

✠

Expanding is the opposite. In expanding, you must be careful to get everything.

Ex 4: *Expand* $(x-2)^2(x+1)$.
Solution:

$$\begin{aligned}(x-2)^2(x+1) &= (x^2-2x-2x+4)(x+1)\\ &= (x^2-4x+4)(x+1)\\ &= x^3-4x^2+4x+x^2-4x+4\\ &= x^3-3x^2+4\end{aligned}$$

✠

Another technique for simplifying things is 'rationalizing the denominator.' If the denominator of a fraction is a square root (or other root), it is nice to put it in the numerator instead. This is how it's done:

Ex 5: *Rationalize the denominators of the following fractions: (a)* $\frac{1}{\sqrt{2}}$*. (b)* $\frac{4x+1}{\sqrt{x}}$.
Solution: (a)

$$\begin{aligned}\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} && \text{We multiply by 1, in such a way that...}\\ &= \frac{\sqrt{2}}{2} && \text{The denominator is not a root.}\end{aligned}$$

(b)

$$\begin{aligned}\frac{4x+1}{\sqrt{x}} &= \frac{4x+1}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}\\ &= \frac{(4x+1)(\sqrt{x})}{x}\\ &= \frac{4x^{3/2}+x^{1/2}}{x}\end{aligned}$$

✠

Exercises

1.1.1: Expand: **(a)** $(z+p)^2$ **(b)** $(x+y)^2$ **(c)** $(x+1)^2$ **(d)** $(x-3)^2$

1.1.2: Expand: **(a)** $(x+y)^3$ **(b)** $(t+4)^3$ **(c)** $(x+\sqrt{2})(x-\sqrt{2})$ **(d)** $(\sqrt{2}+x)(\sqrt{2}-x)$

1.1.3: Factor: **(a)** x^2-4x+3 **(b)** x^2+5x+4 **(c)** x^2-5x+6 **(d)** $2x^2+5x-3$

1.1.4: Factor: **(a)** x^2-16 **(b)** $9-x^2$ **(c)** x^2+2x-3 **(d)** $3x^2+18x+24$

1.1.5: Factor: **(a)** $5x^2-18x-8$ **(b)** x^2-6x+9 **(c)** x^2-2x+1 **(d)** x^2+x-6

1.1.6: Factor and solve for x: **(a)** $x^2-10x-24=0$ **(b)** $x^2+10x-24$ **(c)** $x^2+11x+24=0$ **(d)** $x^2-x-20=0$

1.1.7: Rationalize the denominators of the following: **(a)** $\sqrt{3}/2$ **(b)** $(4x+2)/\sqrt{x+y}$ **(c)** $\frac{x}{\sqrt{2x+1}}$ **(d)** $\frac{y^2-t}{8}$

1.1.8: (NECTA 2006) Simplify: (3 marks)

$$\frac{x}{y^{\frac{1}{2}} + x^{\frac{1}{2}}} + \frac{x}{y^{\frac{1}{2}} - x^{\frac{1}{2}}}\,.$$

1.2 Functions

There are relations, and there are functions. Any equation involving two variables is a relation between them. We are more concerned with functions.

Definition: *A* function *is a relation that is many-to-one or one-to-one.*

• $\mathbb{N}$ote $\mathcal{O}$n $\mathcal{N}$otation •
We can write a function in several ways. The most common is

$$f(x) = 3x + 1,$$

which we read 'f of x is equal to 3 x plus 1.' But we can also write this as

$$f : x \mapsto 3x + 1,$$

which is read 'f maps x to 3 x plus 1.' They mean the same thing. It is uncommon, but also possible, to write the function as a set relation:

$$f = \{(x, y) :\ y = 3x + 1\},$$

which we read 'f is equal to the set of ordered pairs (x, y), such that y is equal to 3 x plus 1.' When writing sets, the description of the set goes inside curly braces { }, and the colon, ':', is read 'such that'.

1.2.1 Domain and Range

There are some things that you can never ever do. Never in this course. A partial list:

- Never divide by 0.
- Never take a logarithm of a negative number or 0.
- Never take a square root of a negative number.

The reason we don't do these things is that they are *not defined*, so it is meaningless to do them. Remembering these is very useful in finding the *domain* of functions.

Definition: *The* domain *of a function is the set of all possible inputs.*

This means that if a function f is a function of x, i.e. $f(x)$, then the *domain of* f is the set of all possible values of x. In most cases, for example $f(x) = 2x + 1$, the domain is 'All real numbers', which means x can be anything. The rules listed above are useful for telling you what values of x are not in the domain in certain cases. Sometimes the domain will be limited, and you will be told explicitly what the domain is.

Ex 1: *Find the domain of* $f(x) = \frac{x}{x-1}$.
Solution: Look at the rules list. There is one rule about division, one rule about logarithms, and one rule about square roots. Which rules apply to our question? Well, we have no logs and no roots, but we do divide. The rule says we cannot divide by 0. What do we divide by? $x - 1$. So, the rule says that $x - 1 \neq 0$. From there it's easy to get that $x \neq 1$.

So 1 is not in the domain, but everything else is. To write this in words, we can say 'The domain is all real numbers except 1.' To write it in math:

$$\text{Domain}(f) = \{x : x \neq 1\}$$

✠

Also note that a positive number raised to any power is always positive. 2^x is always positive, for any x whatsoever, because 2 is positive. In the same way, e^x is always positive, because $e = 2.71828\ldots > 0$.

Can you think of anything else that is always positive? Well, there are square roots, for one. The notations $\sqrt{x}$ is defined as positive. It's true that if $x^2 = 9$ then x can be either 3 or -3, and that is why to correctly solve that equation, you should write $x = \pm\sqrt{9} = \pm 3$. But if the $\pm$ is not there, then it is positive. Also, absolute values, like $|x|$, are always positive.

These 'always positive' things are useful for finding the *range* of functions.

Definition: *The* range *of a function is the set of all possible outputs.*

The range is the set of all possible values of $f(x)$ or of y, depending on how the function is defined. Just like domain, often the range is 'All real numbers.' If we write $y = 2x + 1$ or $f(x) = 2x + 1$ then it is possible for y or $f(x)$ to be anything, so the range of f is all real numbers. However, if we see one of the 'always positive' things, then maybe the range is smaller.

Ex 2: *Find the range of f if $f(x) = x^2 + 1$.*

Solution: What do we have here? We know that $x^2 \geq 0$, so we start with this and turn it into $x^2 + 1$, one step at a time.

$$\begin{aligned} x^2 &\geq 0 \\ x^2 + 1 &\geq 0 + 1 \\ x^2 + 1 &\geq 1 \\ f(x) &\geq 1 \end{aligned}$$

So the range of f is all real numbers greater than or equal to 1. To write it in math,*kwa kihisabati*:

$$\text{Range}(f) = \{y : y \geq 1\}$$

✠

If the domain is limited or restricted, and is a small set, the best way to find the range is just to find where each domain element is mapped. This set is your range.

Ex 3: *Find the range of $f : x \mapsto 3x + 1$ if the domain is $\{-1, 0, 5, 10\}$.*

Solution:

$$f(-1) = -2, \quad f(0) = 1, \quad f(5) = 16, \quad f(10) = 31$$

Thus the range is $\{-2, 1, 16, 31\}$. ✠

1.2.2 Composition of Functions

Definition: Composition of functions *is when you have one function* of *another function. Different from multiplication, an example would be* $\cos(x^2)$*, where you take the cosine of x squared.*

Ex 4: *If $f(x) = x + 2$ and $g(x) = x^2$, find **(a)** $f(g(x))$ and **(b)** $g(f(x))$.*

Solution: This **is not** $f(x) \cdot g(x)$! This **is** f of g of x. And this is how we do it:

(a)

$$\begin{aligned} f(g(x)) &= g(x) + 2 && \text{Start by doing the outer function on the inner function.} \\ &= x^2 + 2 && \text{Then substitute in the inner function.} \end{aligned}$$

(b)

$$\begin{aligned} g\big(f(xbigr) &= \big[f(x)\big]^2 && \text{Same as above.} \\ &= [x+2]^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

✠

Ex 5: *If $f(x) = \cos x$ and $g(x) = x^2 + 1$, find **(a)** $f(g(x))$ and **(b)** $g(f(x))$.*
Solution: **(a)** $f(g(x)) = \cos(g(x)) = \cos(x^2 + 1)$.
(b) $g(f(x)) = [f(x)]^2 + 1 = \cos^2 x + 1$. ✠

1.2.3 Inverses

Definition: *A function $f(x)$ has an* inverse, *written $f^{-1}(x)$, which 'un-does' f, such that*

$$f^{-1}(f(x)) = x.$$

To find the inverse of a function algebraically there is an easy procedure:

1. Write $f(x) =$ as $y =$ (if necessary).
2. Switch x and y everywhere they occur.
3. Make y the subject.
4. Write $f^{-1}(x)$ for y (if Step 1 was necessary).

Ex 6: *Find the inverse of $f(x) = 4x - 8$.*
Solution:

$$\begin{aligned} y &= 4x - 8 && \text{1. Write } y \text{ for } f(x). \\ x &= 4y + 8 && \text{2. Switch } x \text{ and } y. \\ x - 8 &= 4y && \text{3. Make } y \text{ the subject...} \\ \frac{x}{4} - 2 &= y \\ f^{-1}(x) &= \frac{x}{4} - 2 && \text{4. Write } f^{-1}(x) \text{ for } y. \end{aligned}$$

✠

Ex 7: *Find the inverse of $f(x) = x^2$.*
Solution:

$$\begin{aligned} y &= x^2 && \text{1. Write } y \text{ for } f(x). \\ x &= y^2 && \text{2. Switch } x \text{ and } y. \\ \pm\sqrt{x} &= y && \text{3. Make } y \text{ the subject.} \\ f^{-1}(x) &= \pm\sqrt{x} && \text{4. Write } f^{-1}(x) \text{ for } y. \end{aligned}$$

If you wonder about the $\pm$, see part about square roots on page 12. ✠

Ex 8: *Find the inverse of* $f(x) = 2x^2 - 1$.
Solution:

$$\begin{aligned} y &= 2x^2 - 1 && \text{1. Write } y \text{ for } f(x). \\ x &= 2y^2 - 1 && \text{2. Switch } x \text{ and } y. \\ x + 1 &= 2y^2 && \text{3. Make } y \text{ the subject...} \\ (x+1)/2 &= y^2 && \\ \pm\sqrt{\frac{x+1}{2}} &= y && \\ f^{-1}(x) = \pm\sqrt{\frac{x+1}{2}} & && \text{4. Write } f^{-1}(x) \text{ for } y. \end{aligned}$$

✠

Geometrically, the inverse of a function is its reflection over the line $y = x$. And the switching of y and x is the heart of it. If a point (a, b) is a point of $f(x)$, then its reverse, (b, a), is a point of the inverse. And for an inverse, the domain and range are switched. The domain of $f^{-1}(x)$ is the range of $f(x)$, and the range of $f^{-1}(x)$ is the domain of $f(x)$.

$\boxed{x^2 + 1 \text{ and it inverse, reflected over } y = x}$

Figure 1.1: A function and its inverse: Reflections over $y = x$

Exercises

1.2.1: ate the domain and range for the following functions:
(a) $f(x) = \frac{1}{x^2} - 1$ **(b)** $g(x) = 4\cos x - 2$ **(c)** $h(x) = \frac{12}{x^2+x-6}$
(d) $f(x) = e^{2x} - 3$ **(e)** $g(x) = |8x - 3|$ **(f)** $h(x) = \sqrt{x-4}$
(t)

1.2.2: Find the inverses of the following functions:
(a) $f(x) = 3x - 6$ **(b)** $f(x) = -\sqrt{x+4}$ **(c)** $y = (x+3)^2$
(d) $y = x^2 + 3$ **(e)** $f(x) = -x/5 + 3$ **(f)** $g(x) = -x^2$

1.2.3: Let $f(x) = 3x + 1$, $g(x) = -x/2 - 4$, and $h(x) = x^2$. Find the following compositions:
(a) $f \circ g$ **(b)** $g \circ f$ **(c)** $f \circ h$
(d) $h \circ f$ **(e)** $g \circ h$ **(f)** $h \circ g$

1.2.4: (NECTA 2006)
(a) Find $f^{-1}(x)$ if $f(x) = e^x$.
(b) Sketch the graph of $y = e^x$ and its inverse using the same xy-plane.
(c) What is your conclusion about the value of $\log_e N$ if $N \leq 0$? (3 marks)

1.2.5: (NECTA 2005) A function f is defined as:

$$f(x) = \begin{cases} x\text{when} & 0 < x \leq 1 \\ x(x-2)\text{when} & 1 < x < 3 \end{cases}$$

Sketch the graph of $f(x)$. (6 marks)

1.2.6: (NECTA 2002) Let g be the function which is the set of all ordered pairs (x, y) such that $g(x) = \sqrt{x(x-2)}$.

(a) Find the domain and range of g.
(b) Draw a sketch of the graph of g. (6 marks)

1.3 Exponents and Logarithms

This is O-Level material that is never covered well enough in O-Level. You should especially pay attention to the definition of the logarithm.

Most topics will be presented, rather than derived, but the rules of exponents and logarithms are easy, and if you understand where they come from it will help you to remember them.

1.3.1 Exponents

Recall the meaning of exponents:

$$a^2 = a \cdot a, \qquad a^3 = a \cdot a \cdot a, \qquad a^4 = a \cdot a \cdot a \cdot a, \ \ldots$$
$$a^n = \underbrace{a \cdot a \cdot \ \ldots \ \cdot a}_{n}$$

Understanding this, we can see that $a^2 \cdot a^3 = \underbrace{a \cdot a}_{a^2} \cdot \underbrace{a \cdot a \cdot a}_{a^3} = a^5$. So we have found a rule:

$$a^x \cdot a^y = a^{x+y}$$

What about $(a^2)^3$? Well, $(a^2)^3 = (a^2) \cdot (a^2) \cdot (a^2 0 = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a^6$. Thus another rule:

$$(a^x)^y = a^{xy}$$

And what if we have more than just a? It's still easy:

$$(bc)^x = \underbrace{bc \cdot bc \cdot \ \ldots \ \cdot bc}_{x} = \underbrace{b \cdot b \ \ldots \ \cdot b}_{x} \cdot \underbrace{c \cdot c \ \ldots \ \cdot c}_{x} = b^x \cdot c^x.$$

The rule:

$$(bc)^x = b^x c^x$$

Negative exponents are just the same as long as we define $a^{-1} = \frac{1}{a}$.

Now we can see that $a^{-2} = a^{-1} \cdot a^{-1} = \frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a^2}$, the rule is

$$a^{-x} = \frac{1}{a^x}.$$

What about a 0 exponent? Well, $0 = 1 - 1$ so $a^0 = a^1 \cdot a^{-1} = a \cdot \frac{1}{a} = 1$. For any a,

$$a^0 = 1.$$

Even $0^0 = 1$!

Our last exponent rule is for non-integer exponents. How about $a^{1/2}$? Once again we can use the other rules to find the answer. Because $a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1$, we can see that $a^{1/2} = \sqrt{a}$. And, in general,

$$a^{1/n} = \sqrt[n]{a}$$

when n is positive. If n is negative, then $a^{1/n} = 1/\sqrt[|n|]{a}$.

And those are the rules of exponents! They should be easier this time. The good thing about learning things twice is that the second time you can understand better, easier, and faster.

1.3.2 Logarithms

Definition:

$$\boxed{y = \log_b x \quad \text{means} \quad x = b^y}$$

Do not forget it. Know it. Use it. This much is nothing special, it is just the definition. On the logarithmic side, b is called the *base* of the logarithm, so you can read it as 'log base b of x,' or as 'log of x base b.' These two equations are the same, just in different forms. We call $y = \log_b x$ *logarithmic form*, and we call $x = y^b$ *exponential form*. It is easy to remember because the base of the logarithm becomes the base of the exponent. We will start with some examples to show how useful this definition is by itself, then we will find some rules that are also helpful.

Ex 1: *(a) Change* $\log_b x = z$ *to exponential form.*
(b) Change $2^5 = 32$ *to logarithmic form.*
Solution: (a) Using the definition, $b^z = x$.
(b) By definition, $\log_2 32 = 5$. ✠

Ex 2: $5^{\log_5 \pi} =?$
Solution: Using just the definition, let $y = log_5 \pi$. Do not be scared to call part of an equation by a new name. Now write $y = \log_5 \pi$ in exponential form: $5^y = \pi$. But what is y? Again, we write $y = \log_5 \pi$. But $5^y = \pi$. Substituting in for y, we get that $5^{\log_5 \pi} = \pi$. And this is exactly the answer we are looking for. ✠

Ex 3: $8^{\log_8 2} =?$
Solution: Using the definition, if we let $y = \log_8 2$ then, changing to exponential form, $8^y = 2$. Therefore $8^{\log_8 2} = 8^y = 2$. ✠

Nice and easy! These first 2 examples are almost the same, so let's write it in variables now.

$$b^{\log_b x} = x.$$

A description in words is that the number $\log_b x$ is the power to which you can raise b to get x. There are just two constraints: both b and x must be greater than 0. Neither b nor x can be 0 or negative. Now we can go even faster:

Ex 4: $23^{\log_{23} 30} =?$
Solution: $23^{\log_{23} 30} = 30$ ✠

It is tedious to always write out a log's base, so for 2 especially common cases we use abbreviations:
• Note $\mathcal{O}$n $\mathcal{N}$otation •
If the base is 10, we just leave it off, thus $\log(x) = \log_{10}(x)$. Logarithms base e are called 'natural logarithms' and written like this: $\ln(x) = \log_e(x)$.

$$\boxed{y = \ln x \quad \text{means} \quad y = e^x}$$

Ex 5: $e^{\ln 8.24} =?$
Solution: $e^{\ln 8.24} = 8.24$. Why? Because $b^{\log_b x} = x$, and $\ln 8.24 = \log_e 8.24$. ✠

And how about $\log_b b$? We know that $b^{\log_b b} = b$, so it must be that

$$\log_b b = 1.$$

Also, let's look at $\log_b b^x = y$. Switching to exponential form, $b^y = b^x$. So of course, $x = y$. Thus $\log_b b^x = x$.

Now let's find some more general rules. $\log_b 1 =?$ From our conclusion above, we know that $b^{\log_b 1} = 1$. Because we know that $b^0 = 1$ (remember the section on exponents?), this shows that $\log_b 1 = 0$ for any $b > 0$.

Now, let's bring is some other operations. What if $b^x = K$ and $b^y = L$. Switching to logarithmic form, $x = \log_b K$ and $y = \log_b L$. Also, if we multiply and use exponent rules, we can say that $KL = b^x b^y = b^{x+y}$. Therefore,

$$\begin{aligned}
&\log_b(KL) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y \\
\text{but}\quad &\log_b(b^x) = x \qquad \text{and} \quad \log_b(b^y) = y \\
\text{so}\quad &\log_b(b^x b^y) = x + y = \log_b(b^x) + \log_b(b^y) \\
\text{thus :}\quad &\log_b(KL) = \log_b(K) + \log_b(L)
\end{aligned}$$

Also related is that $\log_b(A^n) = \underbrace{\log_b A + \log_b A + \cdots + \log_b A}_{n}$ so $\log_b(A^n) = n \log_b A$. In the same manner it is easy to show that

$$\log_b(A/C) = \log_b A - \log_b C$$

and that

$$\log_b A^{-1} = -\log_b A.$$

Logarithms are also used for simplifying complicated products and quotients, or if there is an unwanted exponent.

Ex 6: *Simplify the following:*

$$100 = \frac{(x^2-3)^4(x+1)^8}{(x-1)^2}$$

Solution: We begin by applying logs to both sides.

$$\begin{aligned}
\log 100 &= \log \frac{(x^2-3)^4(x+1)^8}{(x-1)^2} \\
2 &= \log(x^2-3)^4 + \log(x+1)^8 - \log(x-1)^2 \qquad \text{Now we can use the logarithm rules to simplify.} \\
2 &= 4\log(x^2-3) + 8\log(x+1) - 2\log(x-1)
\end{aligned}$$

And that is far enough. Basi. ✠

Ex 7: *The equation for radioactive decay is $N = N_o e^{-\lambda t}$. Make t the subject.*

Solution: We begin taking logs of both sides. Because there is an e we will use natural logarithms.

$$\begin{aligned}
\ln N &= \ln\big(N_o e^{-\lambda t}\big) \\
\ln N &= \ln N_o + \ln e^{-\lambda t} \\
\ln N &= \ln N_o + -\lambda t \ln e \\
\ln N &= \ln N_o - \lambda t \\
\lambda t &= \ln N_o - \ln N \lambda t \qquad\qquad = \ln\Big(\frac{N_o}{N}\Big) \\
t &= \frac{\ln\big(\frac{N_o}{N}\big)}{\lambda}
\end{aligned}$$

✠

The last formula for logarithms we will not derive, just present. If you are interested in the derivation, please do ask your teacher or find a bigger textbook (or even try it yourself!), it's not too difficult. But here is the **Change of Base Formula:**

$$\log_{b_1} x = \frac{\log_{b_2} x}{\log_{b_2} b_1} \qquad \text{or} \qquad \log_{b_2}(b_1) \cdot \log_{b_1}(x) = \log_{b_2}(x)$$

Ex 8: *Change* $\log_{16} 123$ *to logarithm base 4.*
Solution: Using the change of base formula:

$$\log_{16} 123 = \frac{\log_4 123}{\log_4 16}$$

And, to simplify a bit, $16 = 4^2$ so $\log_4 16 = \log_4 4^2 = 2 \cdot \log_4 4 = 2 \cdot 1 = 2$. Therefore $\log_{16} 123 = \frac{\log_4 123}{2}$. ✠

Usually, if you have a log of a strange base, you will want to change it either to base 10 or base e, because those are the logs that your calculator can do. Look for the LOG and LN buttons on your calculator, learn how to use them. Also find the e^x button. On some calculators it will be labeled EXP.

Ex 9: *Change* ***(a)*** $\log_7 6$ *and* ***(b)*** $\log_{11} 3$ *to both base 10 and base e.*
Solution: **(a)** First let's do base 10. Just apply the change of base formula, with $b_1 = 7$, $b_2 = 10$, and $x = 7$.

$$\log_7 6 = \frac{\log 6}{\log 7}.$$

For base e, it's just the same.

$$\log_7 6 = \frac{\ln 6}{\ln 7}$$

And, these do not contradict each other, if you use a calculator you will find that

$$\log_7 6 = \frac{\log 6}{\log 7} = \frac{\ln 6}{\ln 7} = 0.920782221$$

(b) Same as part (a):

$$\log_{11} 3 = \frac{\log 3}{\log 11} = \frac{\ln 3}{\ln 11} = 0.45815691$$

✠

Logs are strange. We have seen that $\log(ab) = \log a + \log b$, but it does *not distribute!* This means that if $y = a + b$, then

$$\log y = \log(a + b) \neq \log a + \log b.$$

This is another common mistake that is so easy not to make. Do not lose points on your NECTA exam for something so silly. As a quick reference, the logarithm rules are summarized at the end of the chapter on page 31.

Exercises

1.3.1: Write the following in logarithmic form:
(a) $x = b^y$ **(b)** $x = 2^4$ **(c)** $8 = 2^n$
(d) $\pi = e^x$ **(e)** $a = 10^z$ **(f)** $ab = (xy)^{n+1}$
(g) $4 = 2^2$ **(h)** $1024 = 2^{10}$ **(i)** $b^0 = 1$

1.3.2: Write the following in exponential form:
(a) $\log_2 4 = 2$ **(b)** $\log_b c = 8$ **(c)** $\ln 2.9 = 1.06471$
(d) $\ln 2.71828 = 1$ **(e)** $\log 4 = x$ **(f)** $\log_g(ab) = 2$

1.3.3: You should also be proficient at using a calculator to take logarithms. Use a calculator (or your brain in some cases, if you can) to compute the following to 3 decimal places. Look for patterns. What characterizes logarithms of numbers less than the base?
(a) $\log 5.286$ **(b)** $\log 0.321$ **(c)** $\log 10000$
(d) $\ln 1$ **(e)** $\ln\left(\frac{1}{2}\right)$ **(f)** $\ln 2$

1.3.4: Solve for x:
(a) $2\log_5 x = \log_5 18 - \log_5 2$ **(b)** $\log_4(x-1) + \log_4(x+2) = 1$
(c) $\log_4 x - \log_4(2x-1) = 1$ **(d)** $3\log_7 2 = 2\log_7 3 + \log_7 x$

1.3.5: Make x the subject: **(a)** $I = I_o e^{-\mu x}$ **(b)** $4y^2 = 10^{2x}$
(c) $A = Pe^{ix}$ **(d)** $4^2 = 3^5 \cdot 6^x$

1.3.6: Expand by taking natural logs of both sides.
(a) $y = \dfrac{(x^2-3)^4(x^3-1)^8}{(x^2+2x+1)^6}$ **(b)** $y = \dfrac{\sin^4(x)\cdot\cos^3(x)\cdot e^{2x}}{(x^3-1)^{3/2}}$

1.3.7: Solve for x:
(a) $\log_3\sqrt{x} = \sqrt{\log_3 x}$ **(b)** $x^{\sqrt{\log x}} = 10^8$ **(c)** $\log_2(\log_2 x) = 3$

1.3.8: Change the following to natural logs (ln):
(a) $\log 82$ **(b)** $\log_3 9$
(c) $\log_2 256$ **(d)** $\log_b x$

1.3.9: Find the domain and range for the following functions:
(a) $f(x) = \frac{2}{x-5}$ **(b)** $f(x) = \frac{2}{x^2-4}$
(c) $f(x) = x^4 + 5$ **(d)** $f(x) = \sqrt{x} - 2$

1.3.10: (NECTA 2008) Find without using tables (or calculators): (2 marks)

$$\frac{\log 3125 - \log 5}{\log 25 - \log 5}$$

1.3.11: (NECTA 2008) Find the minimum natural number N which satisfies the inequality $0.4^N < 0.001$. (4 marks)

1.3.12: (NECTA 2008) **(a)** Evaluate without using tables or calculators:

$$\log_{10}\left(\frac{1}{3}+\frac{1}{4}\right) + 2\log_{10} 2 + \log_{10}\left(\frac{3}{7}\right).$$

(b) Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find the values of:
(i) $\log_{10} 12$.
(ii) $\log_{10} 5$. (6 marks)

1.3.13: (NECTA 2006) Solve for the real number x if $\log_{10} x = \log_5 2x$. (3 marks)

1.3.14: (NECTA 2005) Solve for x: $4\log_a\sqrt{x} - \log_a 27x = log_a x^{-2}$. (3 marks)

1.3.15: (NECTA 2005) Find y in terms of x: $2\ln y - 3\ln x^2 = \ln\sqrt{x} + \ln x$. (2 marks)

1.3.16: (NECTA 2003) Solve for x in the equation $(\log_x 3)^2 + 3\log_x 3 - 4 = 0$. (2 marks)

1.3.17: (NECTA 2003) Simplify

$$\frac{27^{n+2} - 6 \times 3^{3n+3}}{3^n \times 9^{n+2}}$$

(2 marks)

1.3.18: (NECTA 2002) Use common logarithms to find the value of the following:

$$\begin{array}{ll} (\mathbf{a}) & \dfrac{(7.04)^2}{(31.7)\sqrt{1.09}} \\ (\mathbf{b}) & (115)^{1/2} + 35.2^{2/3} \end{array}$$

(*Note:* It is not apparent why logs are necessary. For part (a) you could use logs and a Four-Figure Table to approximate. For part (b), because it is addition, logarithms don't help. You are better off just evaluating with a calculator.) (6 marks)

1.3.19: (NECTA 2002) Solve the following equations:

$$\begin{array}{ll} (\mathbf{a}) & \log_x 3 + \log_x 27 = 2 \\ (\mathbf{b}) & \log_3 x + 3\log_x 3 = 4 \end{array}$$

(6 marks)

1.4 Coordinate Geometry

Cartesian Coordinates are used very often for describing functions: lines, parabolas, etc. They are named after a French mathematician and philosopher, Rene Descartes. A point, called the origin, is picked, and then we count from the origin a horizontal distance x and a vertical distance y. Any point is called by its horizontal and vertical distances from the origin, denoted (x, y).

A function or a relation is an equation or description of a set of points. A straight line, like $y = 2x + 3$ is a relation between y and x, and the line is all the points for which $y = 2x + 3$ is true.

1.4.1 Slope

Definition: *The* slope *of a line, usually called m is a measure of how steep it is. It is defined as the ratio of vertical distance over horizontal distance between 2 point on the line, often called 'rise over run'.*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Functions of the form $y = mx + b$ are straight lines. They have constant slope, m, and b is the *y-intercept*, the y-value when x is 0. This way of writing the equation, $y = mx + b$, is called *slope-intercept form*, because to write it you must know the slope and the y-intercept of the line.

If the slope is 0, then the line is horizontal. If the slope is undefined, then the line is vertical. Vertical lines are written $x = c$, where c is a constant. This slope formula is only good for straight lines. In Chapter 2 you will learn how to find slope for other functions, like parabolas. There are other ways to write a straight line. Perhaps the easiest way, called *point-slope form* says that the equation of a line with slope m passing through a point (x_o, y_o) is

$$y - y_o = m(x - x_o).$$

Point-slope form is nice because if you know slope and any other point, you can write the equation. It works because for any other point on the line, (x, y), the slope between that point and the point you know, (x_0, y_0) is

$$m = \frac{y - y_o}{x - x_o} \quad \Rightarrow \quad y - y_o = m(x - x_o).$$

Ex 1: *Write the equation for the line of slope 3 passing through the point* $(8, 4)$ ***(a)*** *in point-slope form,* ***(b)*** *in slope-intercept form, and* ***(c)*** *in the form* $y + Ax + B = 0$.

Solution:

(a) For point-slope form, it is too easy. $y - 4 = 3(x - 8)$.

(b) For slope-intercept form, we know it will be $y = 3x + b$, we just need to find b. But, we do know that $(8, 4)$ is point on the line, so we substitute these values in and solve for b:

$$\begin{aligned} y &= 3x + b \\ 4 &= 3 \cdot 8 + b \\ 4 - 24 &= b \\ -20 &= b \end{aligned}$$

So, in slope-intercept form, $y = 3x - 20$.

Another way to find slope-intercept form is to take the point-slope form from (a) and make y the subject:

$$\begin{aligned} y - 4 &= 3(x - 8) && \text{From part (a) above.} \\ y - 4 &= 3x - 24 \\ y &= 3x - 20 \end{aligned}$$

(c) To find this last form, we just take the slope-intercept form and put everything on one side:

$$\begin{aligned} y &= 3x - 20 \\ y - 3x + 20 &= 0 \end{aligned}$$

✠

The downside to point-slope form is that for every point on the line, you can write an equation for the line that looks different, but is really the same as all the others. In slope-intercept form, because the line has only 1 slope and only 1 y-intercept, there is only 1 equation, it is unique.

Two lines with the same slope are parallel, they either are the same at every point, or they never intersect. All other lines will intersect at exactly one point. You can find the point of intersection by setting the y's equal to each other and solving. (Or any method for simultaneous linear equations.)

If two lines have slope m_1 and m_2, and $m_1 \cdot m_2 = -1$, then the lines are perpendicular. 'Normal' is another word that, when used in maths, also means perpendicular.

Ex 2: *What is the slope of a line normal to the line* $y = 4x - 18$*?*

Solution: The slope of our first line is $m_1 = 4$. We need to find a slope m_2 such that $m_1 \cdot m_2 = -1$, or $-1/m_1 = m_2$, so $m_2 = -1/4$. ✠

1.4.2 The Pythagorean Theorem

The Pythagorean Theorem is extremely useful in basic geometry and many other problems. This section has 2 goals: to teach you a very easy proof of the theorem, and then we will see how it becomes the distance formula.

Theorem (Pythagorean Theorem). *If a right triangle has sides of length a, b, and c, where c is the hypotenuse, then*

$$a^2 + b^2 = c^2.$$

Proof: Consider a square of side-length $a + b$, made from triangles, as pictured: What is

$$\boxed{\text{Pythagorean Proof Picture}}$$

the area of the square? There are two good ways to calculate the area. The first is just the normal formula for the area of a square, $A = (a + b)^2$. The second way is to add up the area of all the small shapes. There are 4 triangles of area $A_\triangle = \frac{1}{2}ab$, and there is one square of side-length c, $A_\square = c^2$, so the area of the whole figure is $A = 4 \cdot \frac{1}{2}ab + c^2$. These are just two different methods of calculating the same thing, so they must be equal. $A = A$!

$$\begin{aligned}
(a+b)^2 &= 4 \cdot \frac{1}{2}ab + c^2 \\
a^2 + 2ab + b^2 &= \frac{4}{2}ab + c^2 \\
a^2 + 2ab + b^2 - 2ab &= 2ab + c^2 - 2ab \\
a^2 + b^2 &= c^2
\end{aligned}$$

■

1.4.3 Distance

We use the Pythagorean Theorem to tell us the direct straight-line distance between two points. If there are two points in the plane, (x_1, y_1) and (x_2, y_2), then the straight line between them makes the hypotenuse of a right triangle. The base of the triangle is the horizontal distance between them, $x_2 - x_1$, and the height of the triangle is the vertical distance, $y_2 - y_1$. So, by the Pythagorean Theorem, the distance between these two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is called the *Distance Formula*.

Ex 3: *Find the distance between* $(-1, 3)$ *and* $(5, 1)$*, and find the point that is halfway between them.*

Solution: To find the distance between them, we just use the formula above:

$$d = \sqrt{(5 - -1)^2 + (1 - 3)^2} \qquad = \sqrt{6^2 + (-2)^2} = \sqrt{36 + 4} \qquad = 6.325$$

The point that is halfway between them has an x-coordinate halfway between -1 and 5, and a y-coordinate halfway between 3 and 1. To find these halfway values, we just average:

$$x = \frac{-1 + 5}{2} = 2 \qquad y = \frac{3 + 1}{2} = 2$$

So the midpoint, the point halfway between them, is $(2, 2)$. ✠

A couple more useful definitions:

Definition: *The* midpoint *of a line segment, or the midpoint between two points, is the point that is halfway between the ends of the line segment, or halfway between the two points.*

The midpoint can be found by averaging the x-coordinates and averaging the y-coordinates. The mid point of (x_1, y_1) and (x_2, y_2) is given by:

$$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Definition: *The* perpendicular bisector *of a line segment is the line that is perpendicular to the line segment and passes through the midpoint of the line segment.*

Ex 4: *Find the perpendicular bisector of the line segment connecting points* $(-3, -2)$ *and* $(-9, 2)$.

Solution: To start, let's find the midpoint of this line segment.

$$x = \frac{-3 - 9}{2} = -6 \qquad y = \frac{-2 + 2}{2} = 0$$

The midpoint is $(-6, 0)$. Next, we should find the slope of the line segment.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - -2}{-9 - -3} = \frac{4}{-6} = -2/3$$

The slope of the line segment $m_1 = -2/3$. The line we are looking for is perpendicular to the segment, so that means it has slope m_2 such that $m_1 \cdot m_2 = -1$. Thus $m_2 = 3/2$. Now we have a slope, $3/2$, and a point, $(-6, 0)$. Thus an equation for the perpendicular bisector, in point-slope form, is

$$y = \frac{3}{2} \cdot (x + 6).$$

✠

1.4.4 Equation of a Circle

What is the equation for a circle? To answer that question, ask another: What is a circle?

Definition: *A* circle *is the set of all points a certain distance, called radius, from one point, called the center.*

Let's try to write an equation for a circle centered at the origin. We need an equation for all points that are distance r from $(0, 0)$. What is the distance from a point (x, y) to $(0, 0)$? Remembering the distance formula, our radius is $r = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$. Thus the equation for a circle centered at the origin is $r^2 = x^2 + y^2$. We will use this again later.

That is all well and good for a circle centered at the origin, but what if we want a different center, a certain point (x_o, y_o)? We have to ask the same question: what is the distance from any point (x, y) to our center, (x_o, y_o)? The answer is about the same, this distance, the radius, is $r = \sqrt{(x - x_o)^2 + (y - y_o)^2}$, so the genera equation for a circle of radius r with center at (x_o, y_o) is given by

$$r^2 = (x - x_o)^2 + (y - y_o)^2.$$

Exercises

1.4.1: (NECTA 2008) (a) A, B, and C are three points such that B is the midpoint of $\overline{AC}$. Given that A is $(-1, 6)$ and B is $(2, 4)$, find the coordinates of C.
(b) Determine the slope of the line perpendicular to the line segment $\overline{AC}$. (2.5 marks)

1.4.2: (NECTA 2008) (a) Show, by using Pythagoras' theorem, that the points $A(1, 6)$, $B(-1, 4)$, and $C(2, 1)$ form three vertices of a right-angled triangle.
(b) Find the equation of the perpendicular bisector to the line segment $\overline{BC}$ in (a) above. (3.5 marks)

1.4.3: (NECTA 2005) The equations of two straight lines L_1 and L_2 are $5x - 8y - 80 = 0$ and $8x + 5y - 128 = 0$, respectively. Show that L_1 and L_2 are perpendicular. (2 marks)

1.4.4: (NECTA 2005) The points $A(p, q)$ and $B(p - 1, q + 2)$ lie on the line $2x - y + 3 = 0$. Find the values of p and q. (2 marks)

1.4.5: (NECTA 2005) The points $P(4, -3)$, $Q(-3, 4)$, $R(-2, 7)$, and S are the vertices of a parallelogram. With the help of the coordinates of the midpoint of the diagonal PR, find the coordinates of point S. (2 marks)

1.4.6: (NECTA 2003) A and B are two points whose coordinates are $(2, 1)$ and $(6, 5)$, respectively. Find the equation of the line meeting AB perpendicularly at its midpoint. (2 marks)

1.4.7: (NECTA 2003) Given the points $A(2, -4)$ and $B(-4, 2)$, find the equation of a line which is a perpendicular bisector of $\overline{AB}$. (2 marks)

1.4.8: (NECTA 2002) Find the equations of the straight lines which pass through the centre of the circle $x^2 + y^2 - 4x - 6y - 5 = 0$ and at the points where the given circle cuts the x-axis (each line at one point). (6 marks)

1.4.9: (NECTA 2001) Find the equation of a circle which circumscribes the triangle with vertices $(1, 0)$, $(2, 1))$ and $(0, 2)$. (4 marks)

1.4.10: (NECTA 2000) Find the centre and diameter of the circle $x^2 + y^2 - 4x + 6y - 3 = 0$. (2 marks)

1.4.11: (NECTA 2000) Find the perpendicular distance from the line $4y = 3x - 4$ to the origin. (4 marks)

1.5 Trigonometry

This material is summarized at the end of the chapter on page 31.

We start with some simple word definitions. *Adjacent* means next to, *kanda ya, au ya jirani.* *Opposite* means away from, on the other side, *mkabala.* We use these as a memory aid for the basic trigonometric functions. Imagine a right triangle with angle θ and sides a adjacent to θ, o opposite from θ, and h the hypotenuse as in the figure We define the trigonometric ratios as

Picture of a Right Triangle aoh

Figure 1.2: Right Triangle aoh

follows:

$$\cos\theta = a/h \qquad \sin\theta = o/h \qquad \tan\theta = \cos\theta/\sin\theta = o/h$$

A story to remember this:

> There once was a math student studying trigonometry. He was walking home from school, and he was very confused because he had just learned all about sine, cosine, and tangent. He was thinking so hard to remember their definitions, that he did not look where he was going, so he hit his toe very hard on a big rock.

'Ouch!' he yelled, and he ran to his neighbor, who is also a doctor. The doctor wanted to tell him that he should 'soak his toe' in some warm water and it would feel better. But this doctor came from a different country, so he spoke in a strange accent. Thus, when the doctor tried to say 'Soak the toe,' what he actually said was 'Soh cah toa.'

The student thought for a moment, and then said 'Aha! I know how to remember trigonometry!' He forgot all about his hurt toe because he was so happy. 'Thank you so much, doctor! Because of you I will surely get Division 1 on my exams!'

The key is in what the doctor said. 'Soh' means sin $= o/h$, 'cah' means cos $= a/h$, and 'toa' means tan $= o/a$. So if you have trouble remembering, just think of 'soh cah toa.'

A common mistake is to forget that $\sin x$ or $\cos x$ is a function. By itself, cos has no meaning. You must always have cosine *of* something, be it x or r or θ or π or $Tanzania$. $\cos x$, fine, sin($), sure, tan($Tanzania$), great! But just cos by itself is very bad.

While we are defining things, we should also include the other 3 trigonometric functions, secant (sec), cosecant (csc), and cotangent (cot). They are defined in terms of the original three:

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sec\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

(Note: Sometimes, $\csc x$ is written as cosec$x$.)

1.5.1 Radians

Definition: Radians *are the SI unit for measuring angles. Radians are defined so that if an angle is θ radians, then the length of the arc produced is $r \cdot \theta$, where r is the radius. In a full circle there are 2π radians, because the circumference is $2\pi r$. Therefore,*

$$2\pi \text{ rad} = 360°.$$

How Radians are Defined

Figure 1.3: Definition of Radian

For a general circle of radius r, the arc length s of a piece of the circle of angle θ is given as

$$\begin{aligned} s &= r\theta && \text{if } \theta \text{ is in radians,} \\ s &= \frac{2\pi r}{360} \cdot \theta && \text{if } \theta \text{ is in degrees.} \end{aligned}$$

Also nice is that 'radians' is not actually a unit, it is just a ratio of lengths (s/r). Most angles in radians will not have 'rad' explicitly written. If an angle has no degree sign (°), then it is in radians. Using the equality above, you can convert from radians to degrees and back again just like any other conversion.

Ex 1: *Convert the following from radians to degrees or from degrees to radians:*
(a) $\theta = 60°$
(b) $\phi = 4\pi$
(c) $\alpha = 180°$
(d) $\beta = 3\pi/2$

Solution:

$$
\begin{aligned}
&(\mathbf{a}) \quad \theta = 60^\circ \frac{2\pi}{360^\circ} = \frac{2\pi}{6} = \frac{\pi}{3} \\
&(\mathbf{b}) \quad \phi = 4\pi \frac{360^\circ}{2\pi} = 2 \cdot 360^\circ = 720^\circ \\
&(\mathbf{c}) \quad \alpha = 180^\circ \frac{2\pi}{360^\circ} = \frac{2\pi}{2} = \pi \\
&(\mathbf{d}) \quad \beta = \frac{3\pi}{2} \cdot \frac{360^\circ}{2\pi} = \frac{3}{4} \cdot 360^\circ = 270^\circ
\end{aligned}
$$

✠

1.5.2 Laws of Sines and Cosines

For any triangle where the sides abc are opposite to the angles ABC, the Law of Sines says that

$$\text{Law of Sines:} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and the Law of Cosines says:

$$\text{Law of Cosines:} \qquad a^2 = b^2 + c^2 - 2bc \cos A$$

The proofs of the Law of Sines and Law of Cosines are not here, but they are not too difficult. Try to prove the Law of Sines, maybe you are able!

Triangle ABC with sides abc

Figure 1.4: A generic triangle.

Exercises

1.5.1: (NECTA 2005) Solve the equation $4\cos\theta - 3\sec\theta = 2\tan\theta$ for $-180^\circ \leq \theta \leq 180^\circ$. (3 marks)

1.5.2: (NECTA 2005) Solve the equation $\cos 40^\circ + x) = \sin(2x - 10^\circ)$ when $0^\circ < x < 90^\circ$. (3 marks)

1.5.3: (NECTA 2002) Eliminate θ from the following: (2.5 marks)

$$
\begin{aligned}
x &= \cos 2\theta + 1 \\
y &= \sin\theta + 1
\end{aligned}
$$

1.5.4: (NECTA 2002) Solve the equation $\sin 2\theta - \sin\theta = 0$ for θ between 0° and 180° inclusive. (3.5 marks)

1.5.5: (NECTA 2001) Eliminate x from $a \sin x = b$ and $\tan x = c$. (2 marks)

1.5.6: (NECTA 2001) Solve the following equation for $0^\circ < x < 360^\circ$: (2 marks)

$$\sin 2x + \cos x = 0$$

1.5.7: (NECTA 2000) Prove that (2 marks)

$$2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

1.5.8: (NECTA 2000)
(a) Find a formula for $\sin 3A$ in terms of $\sin A$.
(b) Using the formula obtained in (a) above, find the exact value of $6\sin 50° - 8\sin^3 50°$. (4 marks)

1.6 The Unit Circle

Unit Circle. What does it mean? You know what a circle is, what is the exact meaning of the word 'unit'? Think of *unity*. Unity means *umoja*, and *unit* just means *moja*, 1. Just like in uniform (everyone wears one kind of clothes), unilateral (one side), unicycle (a bicycle with only one wheel), and unicorn (a fantasy horse with a single horn). So the unit circle is just a regular circle of radius 1. But from such a simple form we can draw some very important conclusions.

Remembering back to Section 1.4, the equation for a circle of radius 1 centered at the origin is just $x^2 + y^2 = 1$.

Picture of the Basic Unit Circle

Figure 1.5: The Unit Circle

Check out this picture of the unit circle. The first thing to notice is that the points $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$ are *not* on the circle. Let's look at a point that *is* on the circle. How about this point (x, y) where the line connecting the point to the origin makes an angle θ with the x-axis. How can we find out what x and y are in terms of θ? Sine and cosine! In terms of θ and radius r, $x = r\cos\theta$ and $y = r\sin\theta$. But we know that $r = 1$ because it's a unit circle! So, for any point on the circle, the coordinates are given by

$$(x, y) = (\cos\theta, \sin\theta)$$

where θ is the angle made with the x axis. In other words, for the unit circle $\cos\theta$ correspond to x which is horizontal distance from $(0, 0)$, and $\sin\theta$ corresponds to y which is vertical distance from $(0, 0)$.

Note: If you need more review about the basic trigonometric functions, look at the previous section for details or, for a summary on page 31.

This works for all values of θ. If θ is positive, you move anticlockwise from the x axis, if θ is negative, you move clockwise. If $\theta > 360° = 2\pi$ radians, then you go around once and keep on going around again. Thus the unit circle can be used like a complete graph of $\cos\theta$ and of $\sin\theta$. If you want $\cos\theta$, just look at the x coordinate. If you want $\sin\theta$, just look at y. A useful fact is that the values of $\cos\theta$ and $\sin\theta$ are always in between -1 and 1. You can see it on the unit circle because the maximum and minimum values for x and y are 1 and -1. So we can write

$$-1 \leq \sin\theta \leq 1 \qquad -1 \leq \cos\theta \leq 1,$$

Which is nice. Let's see what else we can find. What did we start with, what's the equation of this circle? $x^2 + y^2 = 1$. And $x = \cos\theta$, and $y = \sin\theta$. So what happens if we substitute in for x and y? We get The Most Important Identity Ever:

$$\boxed{cos^2\theta + \sin^2\theta = 1}$$

This identity is used all the time. Also, think about some angle θ along with $-\theta$. Starting at the x-axis, if you go up by some angle and also down by the same angle, the x coordinates are the same, but the y coordinates are opposite. This means that

$$\cos(-\theta) = \cos\theta \qquad \text{and} \qquad \sin(-\theta) = -\sin\theta.$$

The Unit Circle also can tell us about radians, which is another unit for measuring angles. Radians are better than degrees in almost every way; their one downside is that you aren't familiar with them already. What is the perimeter of the Unit Circle? Probably you remember the formula for perimeter, $C = 2\pi r$. Here, $r = 1$ so the perimeter is 2π. And how many radians are in a full circle? 2π!

As we saw in the previous previous section, if θ is in radians, the arc-length s of a section of angle θ of a circle of radius r is given by $s = r\theta$. For the unit circle, $r = 1$ so $s = \theta$!

A very easy thing that you can do to enable yourself to do problems much faster is to memorize the sines and cosines of all the basic angles, both in radians and degrees. Basic angles are the multiples of $\pi/6$ and $\pi/4$ radians, ($30°$ and $45°$, respectively). If you completely know the full unit circle, as pictured, then you will be able to do many problems quickly in your head, instead of slowly with a calculator.

Picture of the Complete Unit Circle

Figure 1.6: The Complete Unit Circle

1.6.1 Trigonometric Identities and Proofs

A complete list of good trigonometric identities is on page 31. As a student of BAM, you should be able to use these identities to prove other identities. However, about 75% of the time, the only identity you need to know is $\cos^2\theta + \sin^2\theta = 1$.

Ex 1: *Prove that* $\csc^2\theta - \cot^2\theta = 1$
Solution: We'll start with

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 && \text{We need a } \cot^2\theta \text{, which is } \cos^2\theta/\sin^2\theta, \\
\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} && \text{so we divide by } \sin^2\theta \\
\cot^2\theta + 1 &= \csc^2\theta \\
1 &= \csc^2\theta - \cot^2\theta && \text{And we get a new identity!}
\end{aligned}$$

✠

For proofs, there is no procedure that will always work. My advice is this:

- Put everything thing in terms of sine and cosine. Tangents, secants, etc, just confuse things.
- If you have two separate terms, combine them. Find a common denominator and put fractions together.
- Expand! If you have $(1 + \sin\theta)(\cos\theta - 1)$, multiply it out, see what it really is.
- Often, in a question that asks 'Show that this = that,' one side is more simple than the other. Take the complicated side and try to make it like the simple side. It is better to simplify something that is complex than to complicate something that is simple.
- Always remember: $\cos^2\theta + \sin^2\theta = 1$.

Ex 2: *Simplify*

$$\frac{\sin\theta}{1 + \cot^2\theta}$$

Solution:

$$\begin{aligned}
\frac{\sin\theta}{1+\cot^2\theta} &= \frac{\sin\theta}{1+\frac{\cos^2\theta}{\sin^2\theta}} && \text{Put everything in terms of sine and cosine.}\\
&= \frac{\sin\theta}{\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}} && \text{Find a common denominator...}\\
&= \frac{\sin\theta}{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}} && \text{and combine terms.}\\
&= \frac{\sin\theta}{\frac{1}{\sin^2\theta}} && \text{Remember } \sin^2\theta+\cos^2\theta=1,\\
&= \sin^3\theta && \text{and there it is.}
\end{aligned}$$

✠

Ex 3: *Eliminate θ from the equations*

$$x = 3\cos\theta, \qquad \textit{and} \qquad y = 5\sin\theta.$$

Solution: For this kind of question, you are given 2 equations with 3 variables, x, y, and θ. The answer is 1 equation with 2 variables: just x and y. (Eliminate is like *kimbiza au fukuza*. We want to chase away the θ.) To do this, we will use the best identity: $\sin^2\theta = \cos^2\theta = 1$. But first we need to make $\sin\theta$ and $\cos\theta$ the subjects of the given data.

$$\begin{aligned}
x &= 3\cos\theta\\
\cos\theta &= x/3 && \text{Now } \cos\theta \text{ is the subject.}\\
y &= 5\sin\theta\\
\sin\theta &= y/5 && \text{And } \sin\theta \text{ is the subject here.}\\
\cos^2\theta+\sin^2\theta &= \left(\frac{x}{3}\right)^2+\left(\frac{y}{5}\right)^2 && \text{Substituting in,}\\
1 &= \frac{x^2}{9}+\frac{y^2}{25} && \text{And now we have 1 equation without } \theta.
\end{aligned}$$

✠

Ex 4: *If $\cos\theta = -2/3$ and θ is in quadrant III, find the exact value of $\sin\theta$ without using a calculator.*

Solution: Without using a calculator! We need, once again, to use the best identity:

$$\begin{aligned}
\sin^2\theta+\cos^2\theta &= 1\\
\sin^2\theta &= 1-\cos^2\theta\\
\sin^2\theta &= 1-\left(\frac{-2}{3}\right)^2\\
\sin^2\theta &= 1-\frac{4}{9}\\
\sin^2\theta &= \frac{5}{9}\\
\sin\theta &= \pm\frac{\sqrt{5}}{3}
\end{aligned}$$

Now we use the other data, that θ is in quadrant III. If you look at the unit circle, you will see that in quadrant III, both sine and cosine are negative. Therefore

$$\sin\theta = \frac{-\sqrt{5}}{3}$$

✠

Ex 5: *Prove that* $(1-\cos A)(1+\sec A) = \sin A\tan A$.
Solution:

$$\begin{aligned}
&(1-\cos A)(1+\sec A) && \\
&\quad= (1-\cos A)(1+1/\cos A) && \text{Start with the more complicated side;}\\
& && \text{put things in terms of sine and cosine.}\\
&\quad= 1+1/\cos A-\cos A-\cos A/\cos A && \text{Expand.}\\
&\quad= 1/\cos A-\cos A+1-1 && \text{Next, combine with a common denominator,}\\
&\quad= \frac{1}{\cos A}-\frac{\cos^2 A}{\cos A} && \\
&\quad= \frac{1-\cos^2 A}{\cos A} && \text{And remember } \sin^2 A+\cos^2 A = 1,\\
&\quad= \frac{\sin^2 A}{\cos A} && \text{So } \sin^2 A = 1-\cos^2 A\\
&\quad= \sin A\frac{\sin A}{\cos A} && \\
&\quad= \sin A\tan A \qquad \blacksquare &&
\end{aligned}$$

✠

• Note $\mathcal{O}$n $\mathcal{N}$otation •
We usually put a filled-in square ■ at the end of a proof to show that we are done. Sometimes this is called a 'booyah box', as in 'Booyah! I finished the proof!'

Ex 6: *Solve the equation* $2\sin^2\theta-\cos\theta = 1$ *for values of* θ *between 0 and* 2π.
Solution: First we will make the substitution $\sin^2\theta = 1-\cos^2\theta$:

$$\begin{aligned}
2\sin^2\theta-\cos\theta &= 1 &&\\
2(1-\cos^2\theta)-\cos\theta &= 1 &&\\
2-2\cos^2\theta-\cos\theta &= 1 &&\\
-2\cos^2\theta-\cos\theta+1 &= 0 && \text{It's a quadratic!}\\
(-2\cos\theta+1)(\cos\theta+1) &= 0 && \text{Factoring the quadratic.}\\
\cos\theta = \frac{1}{2} \qquad \text{or} \qquad \cos\theta &= -1 &&
\end{aligned}$$

Use the unit circle to find all possible values for θ. For $\cos\theta = \frac{1}{2}$, we find $\theta = \pi/3$ or $\theta = 2\pi/3$. For $\cos\theta = -1$, $\theta = \pi$. Thus $\theta = \pi/3$, $2\pi/3$, or π. ✠

Exercises

1.6.1: (NECTA 2005) Show that

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$$

(3 marks)

1.6.2: (NECTA 2005) Prove that $(\sin\theta + \csc\theta)^2 = \sin^2\theta + \cot^2\theta + 3$. (3 marks)

1.6.3: (NECTA 2003) Solve for x in the equation $\tan 6x = \frac{1}{\sqrt{3}}$ where x is between $-180°$ and $180°$. (2 marks)

1.7 Chapter Revision

Trigonometry

For a right triangle with legs o opposite angle θ, a adjacent to angle θ, and hypotenuse h,

$$\cos\theta = \frac{a}{h} \qquad \sin\theta = \frac{o}{h} \qquad \tan\theta = \frac{o}{a}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{h}{a} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{h}{o} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{a}{o}$$

The Most Important Identity in the World:

$$\sin^2 x + \cos^2 x = 1$$

The identity above, as well as the next three, are all easily derived from the Unit Circle.

$$\cos(-x) = \cos x \qquad \sin(-x) = -\sin x \qquad \tan(-x) = -\tan x$$

Some less important, but still good, identities:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\sin^2\left(\frac{1}{2}\theta\right) = \frac{1-\cos\theta}{2} \qquad \cos^2\left(\frac{1}{2}\theta\right) = \frac{1+\cos\theta}{2}$$

Logarithms

$$\log_b x = y \Leftrightarrow b^y = x \qquad \ln x = y \Leftrightarrow e^y = x$$

$$\log(ab) = \log a + \log b \log(\frac{a}{b}) = \log a - \log b \qquad \log(a^n) = n\log a$$

Change of Base Formula :

$$\log_{b_2} b_1 \cdot \log_{b_1} x = \log_{b_2} x$$

$$\log_{b_1} x = \frac{\log_{b_2} x}{\log_{b_2} b_1}$$

Exercises

1.7.1: (NECTA 2008) Prove that $\csc 2\theta + \cot 2\theta = \cot\theta$. (2.5 marks)

1.7.2: (NECTA 2006) Find an equation (in the form $Ax+By+C=0$) of the line which passes through the point $(2,-1)$ and through through the point of intersection of the line $3x-7+7=0$ and $10x-7y+38=0$. (3 marks)

1.7.3: (NECTA 2006) Find the equation of the perpendicular bisector of the line joining points $A(2,-3)$ and $B(6,5)$.

1.7.4: (NECTA 2006) Using the same xy plane, sketch the graphs of $f : x \mapsto 5 - x$ and $g : x \mapsto x$, and hence calculate the area of the triangle enclosed by the two graphs and the x-axis. (3 marks)

1.7.5: (NECTA 2006) Given that $2A + B = 45°$, show that:

$$\tan B = \frac{1 - \tan^2 A - 2\tan A}{1 - \tan^2 A + 2\tan A}\,.$$

Hence, find the value of $\tan(-15°)$ without using a calculator or mathematical tables. Simplify your answer, and rationalize the denominator. (6 marks)

1.7.6: (NECTA 2003) The straight line $x - y - 6 = 0$ cuts the curve $y^2 = 8x$ at P and Q. Calculate the length of PQ. (2 marks)

1.7.7: (NECTA 2003) Show that

$$\frac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$$

(2 marks)

1.7.8: (NECTA 2003) Simplify

$$\frac{x}{y^{1/2} + x^{1/2}} + \frac{x}{y^{1/2} - x^{1/2}}$$

(2 marks)

1.7.9: (NECTA 2003) Find the set of values of x such for which

$$\frac{4x + 8}{x - 1} > 3.$$

(3 marks)

1.7.10: (NECTA 2001) Find the set of values of x for which

$$\frac{2x + 3}{2x - 1} > 5.$$

(3 marks)

Chapter 2

Differential Calculus

Differentiation finds the exact rate of change of a function. The concept of a 'derivative', 'gradient', 'slope', 'rate of change', are all about the same. Finding the slope of a straight line is easy because it is constant. The slope is the same at any two points on the line. But a function just slightly more complicated, a parabola, has a slope that is different at every point. But with derivatives, we can find what it is at any point. To find this exact slope, we will need to learn limits, then we will learn to differentiate.

Derivatives are incredibly useful. You will take lots and lots of derivatives, you should practice until you can do it quickly and easily. When beginning, show lots of work so that you do not make mistakes, but with practice you will be able to skip some steps, doing them in your head.

2.1 Limits

A limit is a description of what a function does as you look closer and closer to a point, without ever looking at the point. A good example is 'at infinity'. You can never put ∞ in an equation and get an answer, but you can find out what a function does as you get close to infinity.

For example, what happens to the value of y as x approaches ∞ if $y = 4 - \frac{1}{x}$? We write

$$\lim_{x\to\infty} 4 - \frac{1}{x}$$

and say 'The limit of $4 - \frac{1}{x}$ as x approaches infinity.' Obviously, as x gets bigger and bigger 4 will remain as 4, but $\frac{1}{x}$ will get smaller and smaller, approaching 0. Thus we can say that $\lim_{x\to\infty} 4 - \frac{1}{x} = 4 - 0 = 4$, or that the limit of $4 - \frac{1}{x}$ as x approaches infinity is 4.

For limits, we never 'arrive,' we just approach. We get closer and closer. This is important because it avoids problems like dividing by 0. But, also realize that the *direction* of approach matters. In our first example, we were approaching infinity. Because it is impossible to be greater than infinity, we were obviously approaching from below infinity. However, if we want to approach something that is not infinite, like 0, then we can choose to approach from below or from above. We denote these with a $+$ to indicate coming from above or a $-$ to indicate coming from below. For example

$$\lim_{x\to 0^+} \frac{1}{x}$$

is the limit of $1/x$ as x approaches 0 from above. So we think: when x is 1, $1/x$ is 1. Then x gets closer to 0, say $x = 1/2$, then $1/x$ is 2. Then, if $x = 1/100$, even closer to 0, then $1/x = 100$. As x approaches closer and closer to 0, we can see $1/x$ getting very big, very positive. And you can imagine, however big a number you can think of, there is some value for x such that $1/x$ is

x	$1/x$
1	1
1/2	2
1/4	4
1/100	100
1/10000	100000

Table 2.1: As x gets close to 0, $1/x$ gets very big

even bigger, so $1/x$ is *unbounded*, its limit is infinity.

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

But what if we come from the other side of 0? If instead the values we pick for x are -1, then $-1/2, -1/4, -1/100000$, then again $1/x$ will get very big, but it will be negative. Thus the limit coming from below is negative infinity.

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Definition: *The* limit *of a function $f(x)$ at a point p is the value that $f(x)$ approaches as x approaches p. The* left-hand limit *denoted by $\lim_{x\to p^-} f(x)$ is the limit as x approaches p from below, i.e. $x < p$. The* right-hand limit *denoted by $\lim_{x\to p^+} f(x)$ is the limit as x approaches p from above, i.e. $x > p$. The actual limit only exists if all of the following are true:*

- $\lim_{x\to p^-} f(x)$*, the left-hand limit, exists.*
- $\lim_{x\to p^+} f(x)$*, the right-hand limit, exists.*
- $\lim_{x\to p^-} f(x) = \lim_{x\to p^+} f(x)$*, the left-hand limit equals the right-hand limit.*

Otherwise the limit does not exist at p.

Some limits are easy and obvious. If a function is *continuous* at the place you are taking the limit, then the limit is just the value of the function. Being continuous generally means you can draw the graph without lifting your pencil from the paper. No holes, no infinities, just a curved (or straight) line. In fact, the precise definition of continuous is that $f(x)$ is continuous at p if the limit of $f(x)$ as x approaches p is $f(p)$.

Ex 1: *What is $\lim_{x\to 3} x^3 - x^2$?*

Solution: This is a nice easy one. $x^3 - x^2$ is just a polynomial, and polynomials are always continuous. This means we can just substitute in 3 for x and get the answer.

$$\lim_{x\to 3} x^3 - x^2 = 3^3 - 3^2 = 27 - 9 = 18$$

✠

Ex 2: *Find $\lim_{x\to 3}(x^3 - x^2)/(x+1)$.*

Solution: This isn't a polynomial, but it is just one polynomial divided by another. Here we can try substitution, and as long as we don't see a problem (like dividing by 0) we should be fine.

$$\lim_{x\to 3} \frac{x^3 - x^2}{x+1} = \frac{3^3 - 3^2}{3+1} = \frac{18}{4} = 4.5$$

✠

Limits work very nicely, you can add, subtract, multiply, divide, and take powers just like normal. Unfortunately, if a function is not continuous at the place where you are taking the limit, things are more complicated.

Ex 3: *Find* $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$.
Solution: Notice what happens if we just try to substitute:

$$\frac{1^2+1-2}{1^2-1} = \frac{0}{0},$$

which is undefined. But, bahati nzuri, if we factor, we can simplify some.

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x} \quad \text{for } x \neq 1$$

Now we can take the limit!

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

✠

Exercises

Evaluate the following limits:

2.1.1: **(a)** $\lim_{x\to 1} x^3 - x^2$ **(b)** $\lim_{x\to 1} \frac{x^3-x^2}{x-1}$
(c) $\lim_{x\to\infty} x^3 - 1^{1000000}x^2$ **(d)** $\lim_{x\to -\infty} x^3 + x^2$

2.1.2: **(a)** $\lim_{y\to 0} 1/y$ **(b)** $\lim_{z\to 0} 1/|z|$
(c) $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$ **(d)** $\lim_{t\to 0} \frac{(x+t)^3-x^3}{t}$

2.1.3: **(a)** $\lim_{x\to 0^+} \sqrt{x}$ **(b)** $\lim_{x\to -1^+} \sqrt[3]{x}$
(c) $\lim_{x\to 0^-} \sqrt{x}$ **(d)** $\lim_{x\to -1^-} \sqrt[3]{x}$

2.2 Derivatives from First Principles

Most of this book is light on theory, but derivatives by definition are important to understand where derivatives come from, and they tend to show up on NECTA exams, so don't skip over this section.

Now we will take our ideas of limits and apply them to slope. What we want to do is find the exact slope at a point. Consider the parabola $f(x) = x^2$. What is the slope at the point $(1, 1)$? Our formula for slope, $m = (y_2 - y_1)/(x_2 - x_1)$ is only good if we have 2 points. What can we do to find the slope at a single point?

The answer is to take a limit. We will pick a point p on the parabola that is close to $(1, 1)$, and then move p closer and closer, taking the limit as the distance between p and $(1, 1)$ approaches 0. And the result of this limit will be the slope at $(1, 1)$.

Let's start with our point p being 0.5 away from x. Since $x = 1$, this means that $p = 1.5$. And let's give a name to our distance from x, something like h. So we are starting with $h = 0.5$. If the x value of p is 1.5, then what is the y value? We know p is on the parabola $f(x) = x^2$, so the y value must be $y = 1.5^2 = 2.25$. Now, what's the slope of the line connecting $(1, 1)$ to $(1.5, 2.25)$? Using the good old slope formula, we get

$$m = \frac{2.25-1}{1.5-1} = \frac{1.25}{.5} = 2.5.$$

Okay, that's a good start, now let's look closer. If we move p even closer to $(1,1)$, say, maybe at $x = 1.25$, so the horizontal distance $h = 0.25$ from p to $(1,1)$. Now our point is $(1.25, 1.25^2)$. Now

$$m = \frac{1.25^2 - 1}{1.25 - 1} = \frac{0.5625}{.25} = 2.25.$$

Even better. We're going to keep getting closer, but we need a better method. Let's do everything in terms of h, the distance from x. Our first point is always $(1,1)$. Our second point is always $\big(x+h, (x+h)^2\big)$. So that means that the slope we calculate is

$$m = \frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

And, since $x = 1$, and we are trying to make h as small as possible, when we approach $h = 0$, we get $m = \lim_{h\to 0} 2x + h = 2x$. So, when $x = 1$, $m = 2 \cdot 1 + 0 = 2$. So this is the exact slope, or the derivative of $y = x^2$ at the point $(1,1)$.

To write an exact definition, if we have a function $f(x)$, and we want to know the slope, we can calculate the slope of a straight line connecting the point $(x, f(x))$ with a point a distance h away. This slope is

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

Then, to find the exact slope, rather than just an approximation, we take the limit as h approaches 0.

Definition: *The* derivative *of $f(x)$ is defined as*

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

What we did above, finding the derivative at a certain point is fine, but it is so much better to do it in general. Let's see how:

Ex 1: *Find the derivative of $f(x) = x^2$.*
Solution: By our definition,

$$\begin{aligned}
\frac{d}{dx} x^2 &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h = 2x.
\end{aligned}$$

Thus $\frac{d}{dx} x^2 = 2x$. Sweet! ✠

• Note On Notation •

There are a lot of ways to write derivatives. It's because they are so useful. The shortest way to write them was created by Newton. He used a tick mark, called a 'prime'. Thus the derivative of y is y', read 'y prime,' the derivative of $f(x)$ is $f'(x)$, read 'f prime of x,' etc. Newton's notation is very easy, but it has limitations. Mostly in that it doesn't tell you which variable you are differentiating with respect to. Sometimes a function will depend on more than one variable, for example both time and distance, or both vertical and horizontal displacement.

Derivatives were created simultaneously by both Isaac Newton, working in England, and Gottfried Leibnitz, working in Germany. Leibnitz created his own notation, called Leibnitz notation, which is more detailed. It is what we used in the example above. It starts with $\frac{d}{dx}$, which is called the *differential operator*, it tells you to take the derivative with respect to x of what follows, just like cos tells you to take the cosine of what follows. But it's flexible. If you want to differentiate with respect to time, t, you write $\frac{d}{dt}$. So, the instruction 'take the derivative of y with respect to x' is written $\frac{d}{dx}y$, which is read 'd dx of y.' Then, once the derivative is taken, we write $\frac{dy}{dx}$, and say 'dy dx,' or maybe $\frac{df}{dt}$ and say 'df dt.'

Newton's notation is short and easy, Leibnitz notation is longer and exact. The one thing you must remember is that it is a notation, not a fraction, you cannon cancel the d's. $\frac{dy}{dx} \neq \frac{y}{x}$. The d means an infinitely small change, so $\frac{dy}{dx}$ really means 'infinitely small change in y divided by infinitely small change in x,' which is exactly what our derivative is, by definition: slope, but the change in x is infinitely small.

Ex 2: *If $y = x^3 - 2x$, find y'.*
Solution: Here $f(x) = y = x^3 - 2x$, so by definition

$$\begin{aligned}
y' &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h) - (x^3 - 2x)}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h - x^3 + 2x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 2 \\
&= 3x^2 + 3x\cdot 0 + 0^2 - 2 \\
&= 3x^2 - 2
\end{aligned}$$

Thus $y' = 3x^2 - 2$. ✠

Ex 3: *Find the derivative of $f(x) = 1/x$.*
Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) && \text{We'll start by factoring out } 1/h, \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x}{x^2+xh} - \frac{x+h}{x^2+xh}\right) && \text{Finding a common denominator,} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x^2+xh}\right) && \text{Combining,} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{x^2+xh}\right) \\
&= \lim_{h\to 0} \frac{-1}{x^2+xh} && \text{Canceling } h \text{ from the front,} \\
&= \frac{-1}{x^2} && \text{And evaluating the limit.}
\end{aligned}$$

✠

Be careful! *Every* time there is an x in $f(x)$, when writing $f(x+h)$, for every x, write $x+h$ instead.

Ex 4: *Find the derivative of $f(x) = \sqrt{x} + 3$.*
Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} + 3 - (\sqrt{x} + 3)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}
\end{aligned}$$

Now we are going to *rationalize the numerator.* When you have square roots added or subtracted, this trick works. It uses the fact that $(a + b)(a - b) = a^2 - b^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} && \text{Canceling } h, \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} && \text{Taking the limit} \\
&= \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2}\, x^{-1/2}
\end{aligned}$$

✠

This is the method of taking derivatives by definition, or by first principles. For the rest of this chapter we will learn faster ways of differentiating. But, never forget where derivatives come from and what they mean: slope, gradient, rate of change. If a question says to find the derivative 'by definition' or 'from first principles,' then you must use this method.

Exercises

Find the derivatives for the given functions from the definition:
2.2.1: $f(x) = x^2 - x$

2.2.2: $g(x) = x^2 + x - 4$

2.2.3: $y = \sqrt{x+1}$

2.2.4: $f(x) = x^3$

2.2.5: $h(x) = x^3 - x^2 + 5$

2.2.6: (NECTA 2006) Given that $f(x) = x^2 - \frac{1}{2}x + 3$, find the value of $f'(x)$ from first principles. (3 marks)

2.2.7: (NECTA 2003) Find $f'(x)$ from first principles, given that $f(x) = x^3 - 3x^2 + x + 2$. (2 marks)

2.3 Derivatives of Polynomials

Okay, we've seen the complicated way to take derivatives. Now for the easy way. We'll start with the *Power Rule.*

$$\textbf{Power Rule}: \quad \frac{d}{dx}\big(x^n\big) = nx^{n-1}$$

Ex 1: *Differentiate the following:*
(a) $y = x^5$ ***(b)*** $f(t) = t^8$ ***(c)*** $g(x) = x^{8921}$ ***(d)*** $y = x^{-1}$

Solution:
(a) $y' = 5x^4$
(b) $f'(t) = 8t^7$
(c) $g'(x) = 8921x^{8920}$
(d) $y' = -x^{-2}$ ✠

Next we have the *Constant Multiple Rule*. For any constant c,

$$\textbf{Constant Multiple Rule}: \quad \frac{d}{dx}\big[cf(x)\big] = cf'(x)$$

Ex 2: *Differentiate the following:*
(a) $y = 3x^5$ ***(b)*** $f(t) = -t^8$ ***(c)*** $g(x) = \pi x^{8921}$ ***(d)*** $y = cx^{-1}$

Solution:
(a) $y' = 3 \cdot 5x^4 = 15x^4$
(b) $f'(t) = -8t^7$
(c) $g'(x) = \pi \cdot 8921x^{8920}$
(d) $y = cx^{-1}$ ✠

$$\textbf{Addition/Subtraction Rule}: \quad \frac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$$

Ex 3: *Differentiate the following:*
(a) $y = x^5 + x^3$ ***(b)*** $f(t) = t^8 - 2t$ ***(c)*** $g(x) = x^{8921} + x^{486} - 3x^2$

Solution:
(a) $y' = 5x^4 + 3x^2$
(b) $f'(t) = 8t^7 - 2$
(c) $g'(x) = 8921x^{8920} + 486x^{485} - 6x$ ✠

These rules make taking derivatives much faster than always using limits. It is not very difficult to prove these rules using limits and the definition of derivative. Basically, because you can multiply limits by constants, or add and subtract limits, you can do the same with derivatives.

The power rule is especially useful, it is good for any power, not just integers. It also shows that the derivative of a constant is 0. Just think of x as x^1, and 1 as x^0, and $\sqrt{x}$ as $x^{1/2}$.

Ex 4: *Differentiate the following:* ***(a)*** $y = x$ ***(b)*** $y = 5$ ***(c)*** $y = \sqrt{x}$ ***(d)*** $y = x^{-3}$

Solution:
(a) $y' = \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^0 = 1$
(b) $y' = \frac{d}{dx}(5) = \frac{d}{dx}(5 \cdot 1) = \frac{d}{dx}(5 \cdot x^0) = 0 \cdot 5x^{-1} = 0$
(c) $y' = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = 1/2x^{-1/2}$
(d) $y' = -3x^{-4}$ ✠

And that gives us our last rule: the *Constant Rule*:
For any constant c,

$$\textbf{Constant Rule}: \quad \frac{d}{dx}\big(c\big) = 0$$

When taking these derivatives, remember the meaning: slope, gradient, rate of change. A common type of problem asks you to find the equation of a line tangent to a curve at a given point. The way to do this is to differentiate to find the slope in general, then evaluate the derivative at the given x to get the slope of the tangent line at that point. Then, you have the slope of the line, and a point, so it is easy to write the equation for the line in point-slope form.

Ex 5: *Find an equation for the line tangent to the curve* $y = x^3 - 2x^2 + 3$ *at the point* $(3, 12)$.
Solution: As explained, we start by differentiating:

$$y' = 3x^2 - 4x + 0$$

Now we put the x value from our point into the derivative to find the slope at that point. $y'(3) = 2 \cdot 3^2 - 4 \cdot 3 = 18 - 12 = 6$. Basi! We have our point:$(3, 12)$ and our slope, 6, so in point-slope form our line is:

$$y - 12 = 6(x - 3).$$

✠

Sometimes one derivative just isn't enough. You can differentiate again and again. If you differentiate twice, you get the *second derivative*, three times and you get the *third derivative*, etc. In general these are called *higher order derivatives.*

• $\mathbb{N}$ote $\mathcal{O}$n $\mathcal{N}$otation •

Higher order derivatives are written easily in Newton's notations, if a function is y, then y' is the first derivative, y'' is the second derivative, y''' is the third derivative...

For Leibnitz notation, higher order derivatives are written as follows:

$$\begin{aligned} \frac{d}{dx}y &= \frac{dy}{dx} && \text{First derivative,} \\ \frac{d}{dx}\left(\frac{d}{dx}(y)\right) &= \frac{d^2y}{dx^2} && \text{Second derivative,} \\ \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(y)\right)\right) &= \frac{d^3y}{dx^3} && \text{Third derivative.} \end{aligned}$$

This makes sense because the dx in the denominator is one part, and the d in the numerator is one part, and there are many of these. But there is only one y, which is the reason that the y is never squared or cubed.

The meaning of derivatives continues to be the same. If you have a function $f(x)$, then $f'(x)$, the first derivative, is the rate of change of $f(x)$. Then, $f''(x)$ is the rate of change of $f'(x)$, and $f'''(x)$ is the rate of change of $f''(x)$. The best way to understand this is to think of physics. If the position of a particle at time t is given by $s(t)$, then $s'(t)$ is the rate of change of position, called velocity. The second derivative, $s''(t)$, is the rate of change of velocity, called acceleration. So, in general, we can write that

$$\begin{aligned} s(t) &= \text{ position,} \\ s'(t) = v(t) &= \text{ velocity,} \\ s''(t) = v'(t) = a(t) &= \text{ acceleration,} \end{aligned}$$

and we can understand the second derivative as the 'acceleration' of the function.

Exercises

If you have trouble with something, try to write it in the form cx^n. Remember that $\frac{1}{x^n} = x^{-n}$, and that $\sqrt[n]{x} = x^{1/n}$.

2.3.1: Find the following derivatives:
(a) $f(x) = x^3 + 1$ **(b)** $g(x) = 2x^3 + 3x^2$
(c) $f(x) = x^2 + x^2$ **(d)** $g(x) = 2x^2$
Notice that $x^2 + x^2$ in (c) is equal to $2x^2$ in (d). The answers are also the same.

2.3.2: Find the first and second derivatives of the following:
(a) $y = x^5 - \frac{1}{6}x^3 + 12$ **(b)** $R(\theta) = 5\theta^2 - 3\theta$
(c) $f(x) = x^{11} - x + 621$ **(d)** $s(t) = 3t - 6$

2.3.3: Find an equation for the line tangent to the graph of the function at the given point.
(a) $f(x) = x^2 + 3$ at $(-1, 4)$ **(b)** $f(x) = x^3 - 2x$ at $(2, 4)$
(c) $f(x) = (x+3)^2$ at $(0, 9)$ **(d)** $f(x) = \sqrt{x}$ at $(4, 2)$

2.3.4: Differentiate the following:
(a) $f(x) = x^{\frac{3}{2}} + 1$ **(b)** $g(x) = \frac{1}{x}$
(c) $f(x) = \sqrt[3]{x}$ **(d)** $g(x) = x^{-4}$

2.3.5: Find all the points (x-values) where there is a horizontal tangent.
(a) $f(x) = x^2 - 4x + 3$ **(b)** $g(x) = \frac{1}{3}x^3 - 3x^2 + 9x - 1$
(c) $f(x) = \frac{1}{3}x^3 + 4x^2 + 12x + 30$ **(d)** $h(x) = 1/x$

2.3.6: (NECTA 2005) Find the points on the curve $y = x^3 + 3x^2 - 6x - 10$ where the gradient is 3. (3 marks)

2.4 Derivatives of Other Functions

A derivative is a demand. The great Professor Weiquing Gu created a story in her childhood to remember derivatives. She says that a differential operator, $\frac{d}{dx}$ is a tough guy going around and fighting with functions. When $\frac{d}{dx}$ meets a function, it hits the function and asks 'What is your derivative? Tell me your slope!' But different functions respond differently to being hit.

A polynomial function is just normal. When you hit a polynomial it becomes weaker, in degree. Thus x^n becomes $nx^n - 1$. For a polynomial, if you hit it enough, it will die, it will become 0. If the power is negative, then it will just continue becoming more and more negative. This is the power rule from the previous section.

Exponential functions, however, are very strong. When you hit an exponential function, e^x, it stays the same, still e^x. You can hit it 1000 times, and it will still be e^x. The function e^x is a a very strong man.

$$\frac{d}{dx} e^x = e^x$$

This is one of the very special things about the number e. It shows up in many different applications such as damped oscillations and radioactive decay (physics), rates of reactions and pH balances (chemistry), concentration of medicine in blood (biology), and constantly compounded interest (economics).

Then there are the trigonometric functions. Their graphs are like snakes, and the functions are tricky like snakes. When you hit $\sin(x)$, it becomes $\cos(x)$, and then when you hit $\cos(x)$ you get $-\sin(x)$. They are tricky like snakes, always dodging you.

$$\frac{d}{dx} \sin(x) = \cos(x)$$
$$\frac{d}{dx} \cos(x) = -\sin(x)$$

Then, there are logarithms. The natural logarithm is like an abused child. When you hit it, it is scared, and it hides under a table.

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Logarithms of different bases (other than e) are a little bit more complicated, and you don't need to know them for BAM. And, just as before, all of these rules can be proved just from the definition of derivative, but some of them are rather difficult. If you're interested, look for a bigger math book or ask your teacher, the proofs are difficult, but understandable.

We should also present the derivative for tangent:

$$\frac{d}{dx}\tan x = \sec^2 x,$$

where $\sec x = 1/\cos x$.

These rules by themselves are not difficult, and don't forget about the Constant Multiple and Addition/Subtraction rules, they all work together.

Ex 1: *What is the derivative of $y = 5e^x$?*

Solution: By the constant multiple rule, $y' = 5e^x$. It doesn't change at all! ✠

Ex 2: *What is the derivative of $f(x) = \ln x + 2\cos x - \pi x^2$?*

Solution:

$$f'(x) = \frac{1}{x} - 2\sin x - 2\pi x$$

✠

And things work just the same with variables other than x.

Ex 3: *Find the derivative of $R(\theta) = \ln\theta^3 - \theta^{-3}$.*

Solution: Remember that $\ln(\theta^3) = 3\ln\theta$ because of the rules of logarithms. Thus

$$R'(\theta) = 3\cdot\frac{1}{\theta} + 3\theta^{-4}.$$

✠

Exercises

Just apply the derivatives of these new functions along with the rules you already learned for polynomials. Differentiate the following:

2.4.1: **(a)** $f(x) = \sin x + 3x^2$ **(b)** $g(x) = \cos x - \ln x$
(c) $h(x) = e^x + 63x - 1$ **(d)** $X(t) = t^{-4} - \ln t$

2.4.2: **(a)** $f(x) = \tan x - \cos x$ **(b)** $g(x) = 4e^x - 4x^5$
(c) $h(x) = \ln(x^2)$ **(d)** $R(\theta) = \sin\theta - 4\cos\theta$

2.4.3: **(a)** $f(x) = e^x + 7\tan x$ **(b)** $g(x) = 3x^2 - 4/x + \ln x$
(c) $h(x) = \sin x + \cos x + \tan x$ **(d)** $F(r) = \frac{-q_1 q_2}{4\pi\epsilon_o r}$

2.5 Product and Quotient Rules

The Product Rule and the Quotient Rule are used when two functions are multiplied or divided. Good examples would be, if we say $u = x^2$, and $v = \cos x$, then we use the product rule to differentiate $u \cdot v = x^2 \cos x$, and the Quotient Rule is used to differentiate $u/v = x^2/\cos x$.

$$\textbf{Product Rule}: \quad \frac{d}{dx}(u \cdot v) = u \cdot v' + v \cdot u'$$

In words, the derivative of the product of two functions is **the first times the derivative of the second, plus the second times the derivative of the first.**

Ex 1: *Find the derivative of $y = x^2 \cos x$.*
Solution: If we let $u = x^2$ and $v = \cos x$, then $y = uv$. Then, by the Product Rule, $y' = uv' + vu'$.

$$\begin{aligned} u &= x^2 & v &= \cos x \\ u' &= 2x & v' &= -\sin x \end{aligned}$$

$$\begin{aligned} y' &= uv' + vu' \\ &= x^2(-\sin x) + \cos x(2x) \\ &= 2x \cos x - x^2 \sin x \end{aligned}$$

✠

Using the Quotient Rule is similar, but the rule is just a little longer.

$$\textbf{Quotient Rule}: \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Ex 2: *Find the derivative of $f(x) = 3x^2/\sin x$.*
Solution: Looking at the top and the bottom, $u = 3x^2$ and $v = \sin x$. Then we get

$$\begin{aligned} u &= 3x^2 & v &= \sin x \\ u' &= 6x & v' &= \cos x \end{aligned}$$

The quotient rule says that $f'(x) = \frac{vu' - uv'}{v^2}$.

$$\begin{aligned} f'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{\sin x \cdot 6x - 3x^2 \cos x}{\sin^2 x} \\ &= \frac{6x \sin x \cdot -3x^2 \cos x}{\sin^2 x} \end{aligned}$$

✠

A good way to remember the Quotient Rule is in a song. We call the numerator 'hi', because it is up high, and the denominator 'lo', because it is down low. And we put a 'd' in front of something to show a derivative. Then,

$$\frac{d}{dx}\left(\frac{\text{hi}}{\text{lo}}\right) = \frac{\text{lo dhi} - \text{hi dlo}}{\text{lo lo}}$$

Or in words: The derivative of **hi over lo** is **lo dhi minus hi dlo over lo lo**. It is very good to use this song because, for the Quotient Rule, order matters. There is a minus in the numerator, and the first term must be lo dhi. With the Product Rule there is only addition and multiplication, order does not matter. But in the Quotient Rule, because of the subtraction, the order does matter.

Ex 3: *Find the derivative of* $y = \frac{4x^3-2x}{x^2+1}$
Solution:

$$\text{hi} = u = 4x^3 - 2x \qquad \text{lo} = v = x^2 + 1$$
$$\text{dhi} = u' = 12x^2 - 2 \qquad \text{dlo} = v' = 2x$$

And then 'lo dhi - hi dlo over lo lo':

$$\begin{aligned} y' &= \frac{(x^2+1)(12x^2-2) - (4x^3-2x)2x}{(x^2+1)^2} \\ &= \frac{(x^2+1)(12x^2-2) - 8x^4 + 4x^2}{(x^2+1)^2} \end{aligned}$$

Of course it is possible to simplify more, but this is enough. Basi. ✠

Exercises

Find the derivatives using the Product Rule:

2.5.1: **(a)** $u = 3x + 4$ **(b)** $v = \sin x$
(c) $y = (3x + 4)\sin x$

2.5.2: **(a)** $u = 8x^3 - x^2 + 1$ **(b)** $v = 2x^2 - x$
(c) $f(x) = (8x^3 - x^2 + 1)(2x^2 - x)$

2.5.3: **(a)** $g(x) = e^x \cos x$ **(b)** $y = \cos x \sin x$
(c) $R = \cos\theta \cdot \tan\theta$ **(d)** $y = 5x^{-3}\cos x$

2.5.4: **(a)** $y = 6x \ln x$ **(b)** $f(x) = (x^3 + 3x - 1)(x^3 + 3x - 1)$
(c) $g(x) = (x + 3)(x - 3)$ **(d)** $h(x) = (\frac{1}{3}x^3 - 2x)\tan x$

2.5.5: Simplify these into a form such that you can use the product rule, then find the derivatives.
(a) $y = \ln x^{2x}$ **(b)** $y = e^(2x)$
(c) $y = \sin(2\theta)$

Use the Quotient Rule to find differentiate the following:

2.5.6: **(a)** $u = 2x^2$ **(b)** $v = 3x + 1$
(c) $y = \frac{2x^2}{3x+1}$

2.5.7: **(a)** $u = \cos\theta$ **(b)** $v = \sin\theta$
(c) $y = \cot\theta$

2.5.8: **(a)** $f(x) = \frac{5x^3-2}{x+1}$ **(b)** $g(x) = \frac{\ln x}{x^3}$
(c) $y = \frac{x^4-4}{x^2+2}$ **(d)** $H = \frac{2y^2}{e^y}$

2.5.9: **(a)** $Q = \frac{1+\sqrt{x}}{x^2-1}$ **(b)** $y = \frac{3x+8}{3x+8}$
(c) $f(x) = \frac{e^x}{\tan x}$ **(d)** $X(t) = \frac{x^3+1}{x^3-1}$

2.5.10: Your answers for (a) and (b) should be the same.
(a) Use the Quotient Rule to differentiate $y = \frac{3x^4-5x+1}{x}$.
(b) Use the Product Rule to differentiate $y = (3x^4 - 5x + 1) \cdot x^{-1}$.

2.5.11: Use $u = \sin\theta$ and $v = \cos\theta$ to prove that $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$.

2.5.12: Use both rules together to find the derivatives:
(a) $y = \frac{(x^2+1)(2x^3-x^2)}{2x-5}$ **(b)** $y = \frac{5xe^x}{2x^2-1}$ **(c)** $y = \frac{3x^2}{\cos x} \cdot \ln x$

2.5.13: (NECTA 2008) Find the gradient of the curve

$$f(x) = \frac{4x^2 + x - 1}{2x}$$

when $x = 1$. (2 marks)

2.6 The Chain Rule

The Chain Rule is the most powerful tool for differentiation. It is used very often. After learning the Chain Rule you will be able to differentiate just about any function you can think of. There are 2 good ways to state the Chain Rule, one using Newton's notation, the other using Leibnitz's notation. They are equivalent. Sometimes you will find it easier to think in terms of one or the other.
Chain Rule:

$$\frac{d}{dx}\left[f(u)\right] = f'(u) \cdot u'$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Before you can use the Chain Rule, you must know *when* to use it. The Chain Rule applies when you have a composite function, that is a function of a function, for example $\cos(x^3)$. When you read this, you say 'Cosine of x cubed.' The *of* is how you know it's a function *of* a function. It is very different from a product, where of course you use the Product Rule. If there is multiplication, use the Product Rule, if there is a function *of* another function, use the Chain Rule.

Ex 1: *Identify the appropriate rule to differentiate the following:*
(a) $y = \cos(\sin(x))$ ***(b)*** $y = \cos x \cdot \sin x$ ***(c)*** $y = \tan(x^2)$ ***(d)*** $y = \tan^2 x$
Solution:
(a) Chain Rule. **(b)** Product Rule. **(c)** Chain Rule. **(d)** Either, because it could be $\tan x \cdot \tan x$ or $(\tan x)^2$, but Chain Rule is better. ✠

That covers when we use the Chain Rule, now let's move on to *how* it is used.

Ex 2: *Differentiate* $y = \cos(x^3)$.
Solution: To use the Chain Rule, we need to find a good u. Most of the time, if you look inside parentheses ( ) you will find a good u. But, u needs to be bigger than just x. Or, we know how to differentiate cos of something, so whatever that something is needs to be our u. In this case, that something is x^3, so we say,
'Let $u = x^3$.'
and we see that now our problem is to differentiate $y = \cos u$.

$$y = \cos u \qquad\qquad u = x^3$$
$$y' = -\sin u \cdot u' \qquad\qquad u' = 3x^2$$

Substituting u back in so that our answer is just in terms of x,

$$\begin{aligned} y' &= -\sin(x^3) \cdot 3x^2 \\ &= -3x^2 \sin(x^3) \end{aligned}$$

✠

• Note $\mathcal{O}$n $\mathcal{N}$otation •

Be careful on with the difference between products and composition (functions of functions).

$$\sin^2 x = (\sin x) \cdot (\sin x) \neq \sin(\sin x)$$

Now look at this table of derivatives, all the rules we have learned so far, but with the Chain Rule explicitly written in to them.

$$\begin{aligned}
&\frac{d}{dx}(k) = 0 \\
&\frac{d}{dx}(u) = \frac{du}{dx} = u' \\
&\frac{d}{dx}(ku) = k\frac{du}{dx} = ku' \\
&\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} = nu'u^{n-1} \\
&\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} = u'e^u \\
&\frac{d}{dx}(\ln u) = 1u \cdot \frac{du}{dx} = \frac{u'}{u} \\
&\frac{d}{dx}(\cos u) = -\sin(u) \cdot \frac{du}{dx} = -\sin u \cdot u' \\
&\frac{d}{dx}(\sin u) = \cos(u) \cdot \frac{du}{dx} = \cos u \cdot u' \\
&\frac{d}{dx}(\tan u) = \sec^2(u) \cdot \frac{du}{dx} = \sec^2 u \cdot u'
\end{aligned}$$

These are fairly straightforward. Notice the u's, the Chain Rule applies to all of these. We'll have some more examples with the Chain Rule.

Ex 3: *Take the derivative of* $y = \sin(x^2 + 1)$.

Solution: Look at our chart. It says that $\frac{d}{dx}(\sin u) = \cos(u) \cdot u'$. So what is our u? It is $x^2 + 1$. Taking the derivative of u, we see that $u' = 2x$. So, $y' = 2x\cos(x^2 + 1)$. ✠

Ex 4: *Find the derivative of* $f(x) = \cos(x^4 + x^2)$.

Solution: Look at the chart. It says that $\frac{d}{dx}(\cos u) = -\sin(u) \cdot u'$. Let $u = x^4 + x^2$. Taking the derivative of u, $u' = 4x^2 + 2x$. So, $f'(x) = -(4x^2 + 2x)\sin(x^4 + x^2)$. ✠

Ex 5: *If* $y = e^{\cos x}$, *what is* y'*?*

Solution: From the chart $\frac{d}{dx}(e^u) = u' \cdot e^u$. Let $u = \cos x$, and then $u' = -\sin x$. Thus $y' = -\sin x \cdot e^{\cos x}$. ✠

Notice that you must differentiate u, but in the solution you must use both the differentiated u *and* the original u. Be careful to look for when you have a function of a function, as in the examples above, and when you have a product of functions, as in the next example.

Ex 6: $f(x) = \ln x \cdot \sin x$. *Find* f'.

Solution: Here we need to use the Product Rule, because there is multiplication. It says that if $f(x) = u \cdot v$ then $f'(x) = uv' + vu'$. Let $u = \ln x$ and $v = \sin x$. Differentiating, $u' = 1/x$ and $v' = \cos x$, so $f'(x) = \ln x \cdot \cos x + \frac{1}{x}\sin x$. ✠

Ex 7: *If* $y = (\sin x)/(x^3 + 2x)$, *what is* y'*?*

Solution: Here we need the Quotient Rule. It states that $\frac{d}{dx}(u/v) = (vu' - uv')/v^2$. On top we have $u = \sin x$ and underneath $v = x^3 + 2x$. Differentiating, $u' = \cos x$ and $v' = 3x^2 + 2$. Thus

$$y' = \frac{(x^3 + 2x)\cos x - 3x^2 + 2\sin x}{(x^3 + 2x)^2}\,.$$

✠

All the rules work like this. The mistake people make is to use the wrong rule. If you just look carefully, it won't be a problem. This next example illustrates when to use the Power Rule versus when to use the Chain Rule. Look at the differences in the way the functions are written.

In the very first example in this section, for the derivative of $y = \tan^2 x$, the solution said that both Product Rule and Chain Rule work, but that the Chain Rule is better. Let's see why:

Ex 8: *Differentiate* $y = \tan^2 x$
(a) *Using the Product Rule.*
(b) *Using the Chain Rule.*
(c) *Now differentiate* $\tan^5 x$.

Solution:

(a) Using the product rule, we begin by writing out that $y = \tan^2 x = \tan x \cdot \tan x$. Now we can let $u = \tan x$ and $v = \tan x$, and the Product Rule states that $y' = uv' + vu'$. Because $u = v = \tan x$, then $u' = v' = \sec^2 x$, so

$$y' = \tan x \cdot \sec^2 x + \tan x \cdot \sec^2 x = 2\tan x \cdot \sec^2 x.$$

(b) Now, using the Chain Rule we will also rewrite. But this time we say $y = \tan^2 x = (\tan x)^2$. Now we can let $u = \tan x$, so $y = u^2$, $y' = 2u \cdot u'$, and $u' = \sec^2 x$. Then, by the Chain Rule

$$y' = 2 \cdot \frac{du}{dx} = 2u\sec^2 x = 2\tan x \cdot \sec^2 x.$$

(c) So far, perhaps it looks about the same to use the Product Rule and the Chain Rule. But if our exponent is higher, then if you use the product rule you have to use it again and again and again. But with the Chain Rule, you only need to apply it once. Just the same as in part (b), $y = \tan^5 x = (\tan x)^5$. Now let $u = \tan x$, so $y = u^5$. Then $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = \sec^2 x$. Then, by the Chain Rule

$$y' = 5u^4 \cdot u' = 5\tan^4 x \cdot \sec^2 x.$$

The Chain Rule also has the advantage because it works for negative and non-integer powers like -4 and $2/3$ for which the Power Rule is useless. ✠

Ex 9: *Differentiate the following:* ***(a)*** $f(x) = \sin^3 x$*,* ***(b)*** $g(x) = \sin(x^3)$.

Solution: In part **(a)** a we have the function $\sin x$ cubed. The power rule will work very well if we let $u = \sin x$, which means $u' = \cos x$. Then we can rewrite $f(x) = u^3$. Differentiating $f'(x) = 3u^2 \cdot u'$. Finally, substituting in, $f'(x) = 3\sin^2 x \cdot \cos x$.

(b) Here we have the function sin of x^3, so we will use the Chain Rule. Let $u = x^3$, so $u' = 3x^2$, and we rewrite $g(x) = \sin u$. From the chart, $g'(x) = \cos u \cdot u'$. Substituting in, $g'(x) = 3x^2\cos(x^3)$. ✠

The lesson for this section is that if you use the right rule, it is easy to get the right answer. Don't rush into problems. First look carefully to decide which rule you need to use. With the right rule you can differentiate almost anything!

Exercises

Find the derivatives of the following using the Chain Rule:

2.6.1: **(a)** $y = (4-2x)^6$ **(b)** $f(x) = \cos(3x^2+x)$
(c) $g(x) = \tan^4(x)$ **(d)** $y = \sqrt{(x^3-x+1)}$

2.6.2: **(a)** $v = 4\cos(3t-6)$ **(b)** $h = \ln(4x^2-x)$
(c) $y = e^{3x+1}$ **(d)** $X = (2+t^2)^4$

2.6.3: **(a)** $y = (x^3-x^2+x)^9$ **(b)** $y = \frac{1}{(3t^2-t)^4}$
(c) $y = \sin(x^2)$ **(d)** $y = \sin^2 x$

2.6.4: **(a)** $y = \sin(x^3)$ **(b)** $y = \sin^3 x$
(c) $y = \sin(x^4)$ **(d)** $y = \sin^4 x$

2.6.5: **(a)** $f(x) = \sqrt[3]{x^3+1}$ **(b)** $g(r) = \ln(r^3+r^2)$
(c) $h(t) = 3e^{2t^2}$ **(d)** $y = \cos(x^2)$

2.6.6: **(a)** $y = \cos(x^2+x)$ **(b)** $f = \cos(\sin(x))$
(c) $g = \cos^2(x^2)$ **(d)** $A(t) = Pe^{it}$

For the following problems, you need to first find if you should use the Chain Rule, the Product Rule, or the Quotient Rule.

2.6.7: **(a)** $y = xe^x$ **(b)** $f(x) = x^2e^x$
(c) $g(x) = xe^{2x}$ **(d)** $h(x) = x^2e^{2x}$

2.6.8: **(a)** $y = (xe^x)^2$ **(b)** $f(x) = \frac{2}{\sqrt{x^2+1}}$
(c) $g(x) = \sin x \cdot \cos x$ **(d)** $h(x) = \sin(\cos(x))$

2.6.9: **(a)** $y = \frac{\sin(2x)}{\cos(x^2)}$ **(b)** $f = \frac{\cos^4 x}{x^2-1}$
(c) $g = \cos^2 x \cdot \sin^2 x$ **(d)** $h = \sin(2x)\tan(x)$

2.6.10: **(a)** $y = \ln\left(\frac{3x^2-1}{x}\right)$ **(b)** $y = \ln\big((x+1)^4 \cdot \cos(x)\big)$

2.6.11: (NECTA 2008) Differentiate with respect to x, $y = \cos^3 x$. (2 marks)

2.6.12: (NECTA 2006) Find $\frac{dy}{dx}$ given that

$$y = \ln\frac{3x-2}{x+1}\,.$$

(3 marks)

2.6.13: (NECTA 2003) Differentiate the following expression with respect to x: xe^{3x}. (1 mark)

2.6.14: (NECTA 2003) Differentiate the following expression with respect to x: $\ln[\sin(2x)]$. (1 mark)

2.6.15: (NECTA 2001) Differentiate with respect to x: $4\sec(\sqrt[3]{x})$. (2 marks)

2.7 Implicit Differentiation

Implicit differentiation isn't really anything new. It's just the Chain Rule and the Product Rule. About the only things you need to know are

1. $\frac{d}{dx}(x) = 1$, and
2. $\frac{d}{dx}(y) = y'$. Therefore:

3. Using the product rule,

$$\begin{aligned}\frac{d}{dx}(xy) &= x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)\\ &= x \cdot y' + y \cdot 1\\ &= x \cdot y' + y\end{aligned}$$

And that's just about it!

Implicit differentiation is useful in those cases where making y the subject is difficult or even impossible. These are called *implicit functions*. For example, think back to Chapter 1 and the Unit Circle. The equation for the Unit Circle is $x^2 + y^2 = 1$. You can make y the subject, but you end up with $y = \pm\sqrt{1 - x^2}$, which would not be fun to differentiate. So instead, if we want to know the slope of points on the unit circle, we use implicit differentiation.

Ex 1: *Find y' for the Unit Circle, $x^2 + y^2 = 1$.*
Solution: Implicit Differentiation usually has 3 steps:

1. You differentiate both sides of the equation with respect to x.

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}1 && \text{Differentiating both sides,}\\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 && \text{Using the Sum Rule and the Constant Rule}\\ 2x + 2yy' &= 0 && \text{Chain Rule!}\end{aligned}$$

2. Collect on one side terms with y'.

$$2yy' = -2x$$

3. Factor and solve for y'.

$$\begin{aligned}y' &= \frac{-2x}{2y}\\ &= \frac{-x}{y}\end{aligned}$$

✠

Let's look closer and the differentiation in the example above. We differentiate both x^2 and y^2 with respect to x. Differentiating x^2 is just as normal, because the variable in the function matches what we're differentiating with respect to. Thus $\frac{d}{dx}x^2 = 2x$.

But when we have $\frac{d}{dx}y^2$, then the variable of differentiation, x, is different from the variable of the function, y. So we use the Chain Rule. In this case, the Chain Rule would tell us that

$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{dy}{dy}(y^2) \cdot y'\\ &= 2y \cdot y'\end{aligned}$$

That's how it goes.

Ex 2: *Find y'.* $2x = xy - \cos y$

Solution:

$$\begin{aligned}
\frac{d}{dx}(2x) &= \frac{d}{dx}(xy) - \frac{d}{dx}(\cos y) && \text{Differentiating both sides,}\\
2 &= xy' + y + \sin y \cdot y' && \text{Product Rule for } xy \text{, Chain Rule for } \cos y\text{,}\\
2 - y &= (x + \sin y)y' && \text{Collecting terms and factoring,}\\
y' &= \frac{2-y}{x+\sin y} && \text{And solving.}
\end{aligned}$$

✠

Ex 3: *Use implicit differentiation to find y' when $y = \cos^{-1} x$.*
• Note On Notation •
Although $\cos^2 x = \cos x \cdot \cos x$, when we say $\cos^{-1} x$ we mean the inverse cosine function, or arccos*, such that $\cos^{-1}(\cos x) = \arccos(\cos x) = x$. For $1/\cos x$ we use $\sec x$.*
Solution: This one we need to start a little differently. We don't know the derivative of $\cos^{-1} x$, so we'll apply cosine to both sides first. At the end, y' is y' however we get it.

$$\begin{aligned}
y &= \cos^{-1} x\\
\cos y &= x\\
\frac{d}{dx}\cos y &= \frac{d}{dx}x && \text{Now we start the normal process.}\\
-\sin y \cdot y' &= 1 && \text{Using the Chain Rule,}\\
y' &= -1/\sin y && \text{And making } y' \text{ the subject.}
\end{aligned}$$

So that's good, but in this case we can do better. We'd really like to know what this is in terms of x, so we need to write $\sin y$ in terms of x. Recall also our first step, that $\cos y = x$. Using our favorite identity,

$$\cos^2 y + \sin^2 y = 1$$

and then solving for $\sin y$,

$$\sin y = \sqrt{1 - \cos^2 y}$$

and substituting in $\cos y = x$,

$$\sin y = \sqrt{1 - x^2},$$

we have what we want, and we can substitute it into our expression from above:

$$y' = \frac{-1}{\sqrt{1-x^2}}.$$

And that is a pretty answer! ✠

Ex 4: *What is the slope of the relation $6 = x^2 y - 2xy$ at the point where $x = 1$?*
Solution: First we find y', then we'll find the slope at the specific point.

$$\begin{aligned}
\frac{d}{dx}6 &= \frac{d}{dx}(x^2 y) - \frac{d}{dx}(2xy)\\
0 &= 2xy + x^2 y' - 2y - 2xy'\\
2xy' - x^2 y' &= 2xy - 2y\\
(2x - x^2)y' &= 2xy - 2y\\
y' &= \frac{2xy - 2y}{2x - x^2}
\end{aligned}$$

So, we have slope, now we need to find what it is at the given point. Using the original equation, when $x = 2$, $6 = 1^2 y - 1 \cdot 2y = y - 2y = -y$. Therefore $y = -6$, and slope, y' is given by

$$y' = \frac{(2 \cdot 2 \cdot -6) + (-2 \cdot -6)}{2 \cdot 1 - 1^2} = \frac{-12}{1} = -12$$

✠

Exercises

Use implicit differentiation to find y':

2.7.1: **(a)** $xy = 1$ **(b)** $x^2y^2 = 1$ **(c)** $x^3y^3 = 1$ **(d)** $x^2y^2 + xy = x$

2.7.2: **(a)** $x = y^2$ **(b)** $x^{-1}y + y^{-1}x = y^2$ **(c)** $(x + y)^2 = 4$ **(d)** $(x + y)^3 = 8$

2.7.3: **(a)** $\cos x = \sin y$ **(b)** $3x^2 + 2xy - y^2 = 1$ **(c)** $y = \sin^{-1} x$ **(d)** $\ln y = \ln \frac{3x^2}{(x-1)^3}$

2.7.4: (NECTA 2008) Find $\frac{dy}{dx}$ when $x^3 + 3xy + y^3 = 6$. (2 marks)

2.7.5: (NECTA 2005) Find $\frac{dy}{dx}$ for the following equation: $x^2 \sin y - y \cos x = 0$. (3 marks)

2.7.6: (NECTA 2002) Find the value of $\frac{dy}{dx}$ at the point $(1, -1)$ if $x^2 - 3xy + 2y^2 - 2x = 4$. (2.5 marks)

2.7.7: (NECTA 2001) Differentiate with respect to x: $\ln(xy^2) - x + y = 2$. (3 marks)

2.8 To the MAX! The First Derivative Test

One of the most common applications of derivatives is finding maximum and minimum values of functions. Look at the the graph of a parabola in Figure 2.1. When does the maximum of occur? More specifically, what is the slope *before* the maximum? It is positive, increasing. What is the slope *after* the maximum? It is negative, decreasing. So, what is the slope exactly at the maximum? It must be 0!

Graph of $y = -(x - 2)^2 + 2$

Figure 2.1: A parabola. Look at the slope at the maximum y-value.

Maximum means *the highest point*. Nothing can be higher. If the slope at the maximum were positive, that would mean $f(x)$ is increasing, so as x increased, $f(x)$ would go higher than the maximum. But you can't go higher than the maximum. Therefore the slope isn't positive at the maximum. Similarly, if the slope is negative, that means that $f(x)$ is decreasing. So if you look back, if you decrease x a little, $f(x)$ will be higher. But again, $f(x)$ cannot be higher than it's maximum! Therefore, at the maximum value, the slope of $f(x)$ is neither positive nor negative.

So what is it? If it's not positive or negative, it's got to be 0. And that's the lesson. Extreme values, both maxima and minima, occur only at the x-value when the gradient/slope/rate of change is 0.

Definition: *The x-values of a function $f(x)$ for which $f'(x) = 0$ or $f'(x)$ does not exist are called* critical points.

We will only look at situations where $f'(x) = 0$, though it it good to know that if $f'(x)$ does not exist it is also a critical point.

Ex 1: *Find the critical points of* $f(x) = 2x^3 - 3x^2 - 12x - 5$.

Solution: To find critical points, we take the derivative, set it equal to 0, and solve for x.

$$\begin{aligned}
f(x) &= 2x^3 - 3x^2 - 12x - 5 \\
f'(x) &= 6x^2 - 6x - 12 && \text{Take the derivative,} \\
0 &= 6x^2 - 6x - 12 && \text{Set it equal to 0,} \\
0 &= x^2 - x - 2 && \text{And solve for } x. \\
0 &= (x-2)(x+1) && \text{Factoring, see Section 1.1.2 for more info.} \\
x &= 2 \text{ or } -1 && \text{These are the critical points.}
\end{aligned}$$

✠

Only at critical points can a function 'turn,' go from increasing to decreasing or from decreasing to increasing. Unfortunately, a function does not turn at every critical point. If the derivative is negative, then 0, then positive, that means that the function has a local minimum and it looks like this: '$\smile$.' If the derivative is positive, then 0, then negative, that means the function has a local maximum and it looks like this: '$\frown$.' (If the critical point is where $f'(x)$ does not exist, then they function may look like $\vee$ or $\wedge$.)

Definition: *We use the term* local *to describe a minimum or maximum because there may be other points that are bigger or smaller, but* locally*, nearby, in the neighborhood, a local minimum or local maximum is the lowest point or highest point, respectively.*

It is also possible for the derivative to be positive, go down to touch 0, but then return to positive-ness (or be negative-0-negative), in which case there is no maximum or minimum.

A function can only change from increasing to decreasing or from decreasing to increasing at a critical point. So all local maxima and local minima occur at critical points. But not every critical point is a local minimum or maximum.

Thus, the way to find out whether a critical point marks a local maximum, a local minimum, or neither is to look at the sign (positive or negative) of $f'(x)$ before and after the critical point, and see if the function it $\smile$ or $\frown$. This is called the *First Derivative Test.*

Ex 2: *Find the critical points of the following functions, and see if they are local maxima, local minima, or neither:*

(a) $f(x) = x^2 + 4x - 3$
(b) $f(x) = -x^4/4 + 27x - 5$
(c) $f(x) = 4x^3 - 9x^2 - 12x + 3$

Solution: **(a)** First, we need to find critical points. To do this we take the derivative, set it equal to 0, and solve for x:

$$\begin{aligned}
f(x) &= x^2 + 4x - 3 \\
f'(x) &= 2x + 4 && \text{Take the derivative,} \\
0 &= 2x + 4 && \text{Set it equal to 0} \\
x &= -4/2 = -2 && \text{And solve for } x
\end{aligned}$$

We have 1 critical point: $x = -2$. Now we need to find out what is the sign of $f'(x) = 2x + 4$ before and after $x = -2$. It often helps to make a small table:

	Before -2	At -2	After -2
x-values	$x < -2$	$x = -2$	$-2 < x$
Sign of $f'(x)$		0	
Slope of $f(x)$		0	
Shape of $f(x)$		—	

To fill it in, we need to find the sign of $f'(x)$ when $x < -2$. Any x value will work. How about $x = -8$ million. When x =-8 million, $2x + 4$ will be negative. That's all we need to do there. And, when $-2 < x$? Let's pick $x = 0$, that's usually an easy one. $f'(0) = 2 \cdot 0 + 4 = 4$, which is positive. Now we can fill in the table:

x-values	$x < -2$	$x = -2$	$-2 < x$
Sign of $f'(x)$	–ve	0	+ve
Slope of $f(x)$	decreasing	0	increasing
Shape of $f(x)$	\	—	/

And now it's pretty clear that we have a $\smile$, a local minimum. To find the actual value of the minimum, we take our critical point, $x = -2$ and point it back in the *original* function, not the derivative. $f(-2) = (-2)^2 + 4 \cdot -2 - 3 = 4 - 8 - 3 = -7$. So we say that $f(x) = x^2 + 4x - 3$ has a local minimum at $(-2, -7)$.

(b) Same procedure:

$$\begin{aligned} f(x) &= -x^4/4 + 27x - 5 & \\ f'(x) &= -x^3 + 27 & &\text{Take the derivative,} \\ 0 &= -x^3 + 27 & &\text{Set it equal to 0} \\ x^3 &= 27 & &\text{And solve for } x \\ x &= 3 & \end{aligned}$$

Again, we have 1 critical point: $x = 3$. Let's make our small table:

x-values	$x < 3$	$x = 3$	$3 < x$
Sign of $f'(x)$		0	
Slope of $f(x)$		0	
Shape of $f(x)$		—	

At the critical point $x = 3$ we already know that the derivative is 0 and that the shape is a horizontal tangent –. We need to find the sign of $f'(x)$ when $x < 3$. Let's choose $x = 0$. $f'(0) = -0^3 + 27 = 27$, so positive. Now for $3 < x$, after 3: Let's pick $x = 4$. $f'(4) = -4^3 + 27 = -64 + 27 = -37$, which is negative. Now we finish the table:

x-values	$x < 3$	$x = 3$	$3 < x$
Sign of $f'(x)$	+ve	0	–ve
Slope of $f(x)$	inc.	0	dec.
Shape of $f(x)$	/	—	\

There is a maximum at $x = 3$. What is it's value? $f(3) = -3^4 + 27 \cdot 3 - 5 = -5$. Thus $f(x) = -x^4/4 + 27x - 5$ has a local maximum at $(3, -5)$.

(c) And again:

$$f(x) = 4x^3 - 9x^2 - 12x + 3$$

$$\begin{aligned} f'(x) &= 12x^2 - 18x - 12 & &\text{Take the derivative,} \\ 0 &= 12x^2 - 18x - 12 & &\text{Set it equal to 0,} \\ 0 &= 2x^2 - 3x - 2 & &\text{Simplify,} \\ 0 &= (2x + 1)(x - 2) & &\text{And solve for } x. \\ x &= 2 \text{ or } \; -1/2 & &\text{These are the critical points.} \end{aligned}$$

This time, we have 2 critical points: $x = -1/2$ and $x = 2$. Because of 2 critical points, our table will be bigger. We need to find the behavior before $x = -1/2$, in between $-1/2$ and 2, and after $x = 2$. It looks like this:

x-values	$x < -1/2$	$x = -1/2$	$-1/2 < x < 2$	$x = 2$	$2 < x$
Sign of $f'(x)$		0		0	
Slope of $f(x)$		0		0	
Shape of $f(x)$		—		—	

At both critical points the derivative is 0 and that the shape is a horizontal tangent. We need to find the sign of $f'(x)$ when $x < -1/2$. Let's choose $x = -1$. Then $f'(-1) = 12 \cdot -1^2 - 18 \cdot -1 - 12 = 12 + 18 - 12 = 18$, so positive. Next, in between $x = -1/2$ and $x = 2$, we will use $x = 0$. $f'(0) = -12$, negative. Last, $2 < x$, let's pick something really big. If x is huge, then $+12x^2$ is much much bigger than $-18x - 12$, because of the squared. So it will be positive.

x-values	$x < -1/2$	$x = -1/2$	$-1/2 < x < 2$	$x = 2$	$2 < x$
Sign of $f'(x)$	+ve	0	–ve	0	+ve
Slope of $f(x)$	inc.	0	dec.	0	inc.
Shape of $f(x)$	/	—	\	—	/

There is a maximum at $x = -1/2$ and a minimum at $x = 2$. $f(-1/2) = 25/4$ and $f(2) = -10$. Thus $f(x) = 4x^3 - 9x^2 - 12x + 3$ has a local maximum at $(-1/2, 25/4)$ and a local minimum at $(2, -10)$. ✠

Just one more example for the case where the critical point is neither a minimum nor a maximum.

Ex 3: *Find the critical points of $y = x^5$ and evaluate whether they are maxima, minima, or neither.*
Solution: Finding critical points:

$$\begin{aligned} y &= x^5 \\ y' &= 5x^4 \\ 0 &= 5x^4 \\ x &= 0 \end{aligned}$$

For $x < 0$ choose $x = -1$. $f'(-1) = 5 \cdot (-1)^4 = 5$ is positive. For $0 < x$ choose $x = 1$, $f'(1) = 5 \cdot 1^4 = 5$ is also positive. The chart then is:

x-values	$x < 0$	$x = 0$	$0 < x$
Sign of $f'(x)$	+ve	0	+ve
Slope of $f(x)$	inc.	0	inc.
Shape of $f(x)$	/	—	/

Because $f(x)$ is never decreasing, there is no maximum or minimum. This critical point is not a turning point, it is just a point where there is a horizontal tangent. ✠

Exercises

2.8.1: Find all critical points of the following functions: **(a)** $y = \frac{2}{3}x^3 - 3x^2 + 4x - 5$ **(b)** $f(x) = 3x^2 - 4$ **(c)** $R(\theta) = \sin\theta$ **(d)** $S(\theta) = \cos\theta$

2.8.2: For each of the functions, find the critical points, then evaluate if they are local maxima, local minima, or neither. **(a)** $X(t) = -4t^2 + 2t$ **(b)** $f(x) = x^3$ **(c)** $y = x^3 - 4x$ **(d)** $R(\theta) = \sin\theta$

2.8.3: (NECTA 2008) A farmer has 120 metres of fencing with which to enclose a rectangular sheep-pen, using a wall for one side. Find that maximum area that can be enclosed. (5 marks)

2.8.4: (NECTA 2005) A farmer encloses sheep in a rectangular field using hurdle for 3 sides and a long wall for the fourth side. If he has 100m of hurdles, find the greatest area he can enclose. (5 marks)

2.8.5: (NECTA 2002) A rectangular block has a square base whose length is x centimetres. Its total surface area is 150 cm^2.
(a) Show that the volume of the block is $\frac{1}{2}(75x - x^3)$ cm^3.
(b) Calculate the dimensions of the block when its volume is maximum. (6 marks)

2.9 Fanya Tena: The Second Derivative Test

The Second Derivative Test does the same thing as the First Derivative Test, and more! It also can do it faster, sometimes. First, we need to define concavity.

Definition: *A function $f(x)$ is called* concave up *on intervals where $f''(x) > 0$, and* concave down *on intervals where $f''(x) < 0$.*

Second derivatives, concavity, there like acceleration. They answer the question 'How is the slope of $f(x)$ changing?' Just like anything else, if $f''(x)$ is continuous, then it can't jump from positive to negative without crossing 0.

Definition: *The x-values of a function $f(x)$ where $f''(x) = 0$ (or does not exist) are called* inflection points. *Functions can change concavity only at inflection points.*

Furthermore, what does $f'(x)$ do around a local maximum? A local maximum means $f'(x)$ changes from positive to negative, which means $f'(x)$ is decreasing. In terms of second derivatives, this means that $f''(x)$ is negative.

By the same logic $f''(x)$ is positive at local minima. So instead of doing a sign chart, all you need to do is look at the sign of the second derivative at a critical point to know if it is a local minimum or a maximum.

The Second Derivative Test:

- If $f'(c) = 0$ and $f''(c)$ is positive, then $(c, f(c))$ is a local minimum.
- If $f'(c) = 0$ and $f''(c)$ is negative, then $(c, f(c))$ is a local maximum.

The only weakness here is that if $f''(x) = 0$ at a critical point, then the test is inconclusive. It might be a local maximum. It might be a local minimum. Or it might be neither. You'll have to use the First Derivative Test.

Ex 1: *Use the Second Derivative Test to find all local extrema for $f(x) = x^2 + 4x - 3$*
Solution: We start out the same old way: find critical points.

$$\begin{aligned} f(x) &= x^2 + 4x - 3 & \\ f'(x) &= 2x + 4 & &\text{Take the derivative,} \\ 0 &= 2x + 4 & &\text{Set it equal to 0,} \\ x &= -4/2 = -2 & &\text{Critical Point.} \end{aligned}$$

Now we need the Second Derivative.

$$f'(x) = 2x + 4$$
$$f''(x) = 2$$

So, that's nice and easy. $f''(x) = 2$ always, so the second derivative is always positive. That means our critical point is a local minimum. Again, to find the actual point, we use the original function. $f(-2) = (-2)^2 + 4 \cdot -2 - 3 = -7$. Thus $f(x) = x^2 + 4x - 3$ has a local minimum at $(-2, -7)$. ✠

Ex 2: *Find local extrema of* $f(x) = \frac{-1}{3}x^3 + 2x^2 + 4x + 1$.
Solution: First we find critical points.

$$\begin{aligned} f(x) &= \frac{-1}{3}x^3 + 2x^2 + 4x + 1 \\ f'(x) &= -x^2 + 4x + 4 && \text{Take the derivative,} \\ 0 &= -x^2 + 4x + 4 && \text{Set it equal to 0} \\ x &= (-x + 2)(x + 2) && \text{Factoring,} \\ x &= -2 \text{ or } 2 && \text{Critical Points.} \end{aligned}$$

Now we need the Second Derivative.

$$f'(x) = -x^2 + 4x + 4$$
$$f''(x) = -2x + 4$$

How does it look at our critical points? When $f''(-2) = 8$, so $f(x)$ has a local minimum at $x = -2$. However, $f''(2) = 0$, so the Second Derivative Test is inconclusive. For this critical point, we have to use the First Derivative Test. When x is between -2 and 2, say, $x = 0$, then $f''(0) = 4$. And when $2 < x$, say $x = 3$, then $f''(3) = -2$.

x-values	$-2 < x < 2$	$x = 2$	$2 < x$
Sign of $f'(x)$	+ve	0	–ve
Slope of $f(x)$	inc.	0	dec.
Shape of $f(x)$	/	—	\

So at $x = 2$, $f(x)$ has a local maximum. $f(-2) = 8/3 + 8 - 8 + 1 = 11/3$ and $f(2) = -8/3 + 8 + 8 + 1 = 43/3$. Therefore $(-2, 11/3)$ is a local minimum and $(2, 43/3)$ is a local maximum. ✠

Exercises

2.9.1: (NECTA 2005) A farmer encloses sheep in a rectangular field using hurdle for 3 sides and a long wall for the fourth side. If he has 100m of hurdles, find the greatest area he can enclose. (5 marks)

2.9.2: (NECTA 2002) A rectangular block has a square base whose side-length is x cm. Its total surface area is 150 cm^2.
(a) Show that the volume of the block is $1/2(75x - x^3)$ cm^3.
(b) Calculate the dimensions of the block when its volume is maximum. (6 marks)

2.10 Applications of Derivatives

Exercises

2.10.1: (NECTA 2003) A hemispherical bowl of radius 6 cm contains water which is flowing into it at a constant rate. When the height of water is h cm, the volume V of water in the bowl is given by

$$V = \pi\left(6h^2 - \frac{1}{3}h^3\right) \text{ cm}^3.$$

(a) Given that $h = 3$, find the rate of change of the volume of water with respect to its level.
(b) What is the rate of change of the volume of water if $h = 3$ and $t = 1$ minute.
(c) Find the rate at which the water level is rising when $h = 3$, given that the time taken to fill the bowl is 1 minute. (6 marks)

2.10.2: (NECTA 2002) Find the value of $\frac{dy}{dx}$ at the point $(2/3, 3, 4)$, if

$$x = \frac{2t}{t+2} \qquad \text{and} \qquad y = \frac{3t}{t+3}$$

(3.5 marks)

2.10.3: (NECTA 2000) A curve is represented parametrically by

$$x = \frac{1}{1+t} \qquad \text{and} \qquad y = \frac{t^3}{1+t}.$$

Find $\frac{dy}{dx}$ in terms of t, then find the gradient to the curve at the point $(x, y) = (1/2, 1/2)$. (4 marks)

2.10.4: (NECTA 2000) Fencing is to be added to an existing wall of length 20m. how should the extra fence be added to maximize the area of the enclosed rectangle if the additional fence is 80m long? Calculate the maximum area. (7 marks)

2.11 Chapter Revision

Chapter Revision

$$\begin{aligned}
&\frac{d}{dx}(k) = 0\\
&\frac{d}{dx}(u) = \frac{du}{dx} = u'\\
&\frac{d}{dx}(ku) = k\frac{du}{dx} = ku'\\
&\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} = nu'u^{n-1}\\
&\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} = u'e^u\\
&\frac{d}{dx}(\ln u) = 1u \cdot \frac{du}{dx} = \frac{u'}{u}\\
&\frac{d}{dx}(\cos u) = -\sin(u) \cdot \frac{du}{dx} = -u' \sin u\\
&\frac{d}{dx}(\sin u) = \cos(u) \cdot \frac{du}{dx} = u' \cos u\\
&\frac{d}{dx}(\tan u) = \sec^2(u) \cdot \frac{du}{dx} = u' \sec^2 u
\end{aligned}$$

Chapter 3

Integral Calculus

3.1 Integrals as Antiderivatives

Definition: Integration *is differentiation backwards.*

$$\int f(x)\,dx = F(x) \quad \text{means} \quad \frac{d}{dx}F(x) = f(x)$$

If the *integral* of $f(x)$ is $F(x)$, then it means that the derivative of $F(x)$ is $f(x)$.
• $\mathbb{N}$ote $\mathcal{O}$n $\mathcal{N}$otation •
In the expression

$$\int f(x)\,dx$$

$\int$ is the *integral sign*, $f(x)$ is called the *integrand*, and dx is the *variable of integration*, or the variable that you are integrating *with respect to*.

For example, when you see $\int 2x\,dx$ the answer is all functions that you can differentiate to get $2x$. So what is the answer? x^2! Because $\frac{d}{dx}x^2 = 2x$. But that's not all. $\frac{d}{dx}(x^2 + 1 = 2x)$, and $\frac{d}{dx}(x^2 + 2 = 2x)$, and even $\frac{d}{dx}(x^2 - \pi^7 + 23 = 2x)$. So there isn't just one answer, there are infinitely many answers. What we do to avoid this problem is to use a constant, c, called the *constant of integration*.

Ex 1: *Evaluate* $\int 2x\,dx$.
Solution:

$$\int 2x\,dx = x^2 + c$$

This is true because whatever the constant value of c, $\frac{d}{dx}(x^2 + c = 2x)$. ✠

Integration is very nice in that you can (and should) **always** differentiate your answer. If you get the integrand, then your answer is right. It is so easy to check that you should never get an answer wrong without knowing that it is wrong.

Ex 2:

$$\int \sin x\,dx$$

Solution: Think, what can you differentiate to get $\sin x$? Of course, it's $\cos x$!, but be careful, positive or negative?

$$\int \sin x\,dx = -\cos x + c$$

Checking the answer: $\frac{d}{dx}(-\cos x + c) = --\sin x = \sin x$, so it's good. ✠

Ex 3:

$$\int 3x^2 + \frac{1}{x}\,dx$$

Solution: What can you differentiate to get $3x^2$? x^3. What about $1/x$? It's $\ln x$. Thus

$$\int 3x^2 + \frac{1}{x}\,dx = x^3 + \ln x + c.$$

Differentiating to check the answer: $\frac{d}{dx}(x^3 + \ln x + c) = 3x^2 + 1/x$. ✠

Exercises

Just think about what these look like the derivatives of. Don't forget about '$+c$'!

3.1.1:

(**a**) $\displaystyle\int 4x^3 + 3x^2\,dx$ (**b**) $\displaystyle\int e^x\,dx$

(**c**) $\displaystyle\int \frac{1}{x}\,dx$ (**d**) $\displaystyle\int \cos x\,dx$

3.1.2:

(**a**) $\displaystyle\int 100x^{99}\,dx$ (**b**) $\displaystyle\int 3x^2\,dx$

(**c**) $\displaystyle\int \frac{3}{2}x^2\,dx$ (**d**) $\displaystyle\int \frac{1}{2}x^2\,dx$

3.1.3: (NECTA 2006) A curve that passes through the origin has a gradient $2x - 1$. Find the equation of this curve in terms of x and y. (2 marks)

3.2 Integration Rules

In the last section we concentrated on the meaning of an integral as an antiderivative. Here we focus on the mechanics of integration. Here is a list of integration rules. They all come directly from differentiation rules.

$$\begin{aligned}
&\int k\,dx = kx + c\\
&\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx\\
&\int kf(x)\,dx = k\int f(x)\,dx\\
&\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \quad \text{for } n \neq -1\\
&\int e^x\,dx = e^x + c\\
&\int \frac{1}{x}\,dx = \ln x + c\\
&\int \cos x\,dx = \sin x + c\\
&\int \sin x\,dx = -\cos x + c\\
&\int \sec^2 x\,dx = \tan x + c
\end{aligned}$$

And now you just need practice at applying them again and again.

Ex 1:

$$\int 3x^8 - \sin x\,dx$$

Solution:

$$\begin{aligned}
\int 3x^8 - \sin x\,dx &= \int 3x^8\,dx + \int -\sin x\,dx\\
&= 3\int x^8\,dx - \int \sin x\,dx\\
&= \frac{3}{9}x^9 - (-\cos x) + c\\
&= \frac{1}{3}x^9 + \cos x + c
\end{aligned}$$

✠

Ex 2:

$$\int 200x^{99} - 30x^5 + 8x\,dx$$

Solution:

$$\begin{aligned}
\int 200x^9 9 - 30x^5 + 8x\,dx &= \int 200x^{99}\,dx - \int 30x^5\,dx + \int 8x\,dx\\
&= \frac{200}{100}x^{100} - \frac{30}{6}x^6 + \frac{8}{2}x^2 + c\\
&= 2x^{100} - 5x^6 + 4x^2 + c
\end{aligned}$$

✠

And so forth. These are not difficult, but you need lots and lots of practice!

Exercises

3.2.1: (NECTA 2008) Integrate the following expression with respect to x: $\sec x$.
Note: This is impossible. The integral $\int \sec x\,dx$ is not defined. You should write this on a NECTA Exam, and maybe instead perform the integral of $\sec^2 x$ with respect to x. (1 mark)

3.2.2: (NECTA 2001) Find y in terms of x given that $\frac{dy}{dx} = x(1-x)$ and that $y = 1/2$ when $x = 0$. (4 marks)

3.3 Area and the Definite Integral

Definition: *If* $\int f(x)\,dx = F(x)$*, then we define the* definite integral *from a to b of $f(x)$ as:*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We call a and b the lower bound *and* upper bound*, respectively, or the* limits of integration.

Ex 1: *Evaluate*

$$\int_2^6 x\,dx$$

Solution:

$$\begin{aligned}
\int_2^6 x\,dx &= \left[\frac{x^2}{2}\right]_2^6 && \text{We integrate like normal, keeping the bounds}\\
&= \frac{6^2}{2} - \frac{2^2}{2} && \text{But then plug in the upper and lower bounds,}\\
&= 36/2 - 4/2 && \text{And subtract the lower from the upper.}\\
&= 18 - 2 = 16
\end{aligned}$$

✠

Ex 2: *Evaluate*

$$\int_0^3 x^2 + 1\,dx$$

Solution:

$$\begin{aligned}
\int_0^3 x^2 + 1\,dx &= \left[\frac{x^3}{3} + x\right]_0^3 && \text{We integrate like normal,}\\
&= \left(\frac{3^3}{3} + 3\right) - \left(\frac{0^3}{3} + 0\right) && \text{But then plug in the upper and lower bounds,}\\
&= 9 + 3 - 0 && \text{And subtract the lower from the upper.}\\
&= 12
\end{aligned}$$

✠

For definite integrals, there is no need for $+c$. Watch what happens if we put it in (using the problem from Ex. 1):

$$\begin{aligned}
\int_2^6 x\,dx &= \left[\frac{x^2}{2}+c\right]_2^6 && \text{If we use } +c, \\
&= \left(\frac{6^2}{2}+c\right) - \left(\frac{2^2}{2}+c\right) && \text{It stays around a little while,} \\
&= 36/2 + c - 4/2 - c && \text{But then goes away in the last step,} \\
&= 18 - 2 = 16 && \text{Leaving the answer with no } c.
\end{aligned}$$

And it will always work that way. It will always go away. So, for definite integrals, we do not use $+c$.

3.3.1 Integrals as Sums

Up until now, we have treated integration only as the opposite of differentiation. Now it is time to learn the true meaning of integration. Integration is a sum. It is a sum of infinitely small

Integration as Area

Figure 3.1: Integration as a sum of rectangles.

little pieces. For example, if look at

$$\int_0^1 x^2\,dx\,,$$

the actual meaning of the integral is to take little rectangles of height x^2 and width dx, multiply to get their area ($x^2\,dx$), and then add them up. Thus the integral above is actually a formula for finding the total area between x^2 and the x-axis from $x = 0$ to $x = 1$. I encourage an interested student to find a more comprehensive book that will prove this.

The area under the curve $f(x)$, between $f(x)$ and the x-axis, from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)\,dx\,.$$

Ex 3: *Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = -1$ to $x = 1$*

Solution:

$$\begin{aligned}
A &= \int_{-1}^1 x^2 + 1\,dx \\
&= \left[\frac{x^3}{3}+x\right]_{-1}^1 \\
&= \left(\frac{1}{3}+1\right) - \left(\frac{-1}{3}-1\right) \\
&= \frac{2}{3} + 2 = \frac{8}{3}
\end{aligned}$$

✠

Ex 4: *Find the area under the curve $f(x) = 8x^3 - 2x$ from 0 to 2.*

Solution:

$$\begin{aligned}
A &= \int_0^2 8x^3 - 2x\,dx \\
&= \left[\frac{8x^4}{4} - \frac{2x^2}{2}\right]_0^2 \\
&= \left[2x^4 - x^2\right]_0^2 \\
&= (2\cdot 2^4 - 2^2) - 0 \\
&= 32 - 4 = 28
\end{aligned}$$

✠

Exercises

3.3.1: (NECTA 2005) Find the area under the curve $y = x^2(x-2)$ from $x = 0$ to $x = 8/3$. (5 marks)

3.3.2: (NECTA 2001)

$$\text{Evaluate} \qquad \int_0^{\frac{x}{a}} \sin ax\,dx$$

(2 marks)

3.3.3: (NECTA 2001) Find the area enclosed by the curve $y = x^3 + 5x^2 + 4x$ and the x-axis between $x = 0$ and $x = 4$. (3 marks)

3.4 U-S(U)bstit(U)tion

In many ways, the U-Substitution in integration is similar to how we used the Chain Rule back in Section 2.6. All of the integration rules we have seen are just as good for u and du as they are for x and dx. For example, we can take the power rule, and change it to use u and du instead of x and dx:

If u is any differentiable function, then
$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + c,$$

as long as $n \neq -1$.

Ex 1: *Find* $\int (x-3)^4\,dx$.

Solution: We cannot integrate this by normal rules, but if we let $u = x - 3$, then $\frac{du}{dx} = 1$, so $du = dx$, and the integral is now

$$\begin{aligned}
\int (x-3)^4\,dx &= \int u^4\,du && \text{Substituting } u = x-3,\ du = dx, \\
&= \frac{1}{5}u^5 + c && \text{Integrating,} \\
&= \frac{1}{5}(x-3)^5 + c && \text{and substituting back in.}
\end{aligned}$$

✠

For example, we know that $\int \cos x\,dx = \sin x + c$, so we can rewrite the rule with u's saying that $\int \cos u\,du = \sin u + c$.

Ex 2: *Find* $\int \cos(x+1)\,dx$.

Solution: Almost always, we look inside parentheses ( ) to find a u. Here, $x+1$ is inside parentheses, so we say

$$\text{Let} \quad u = x+1.$$

Now we need differentiate to find du.

$$\begin{aligned} u &= x+1 \\ \frac{du}{dx} &= 1 \\ du &= dx \end{aligned}$$

That's good. Now we can substitute our integral. $u = x+1$ and $du = dx$, so

$$\begin{aligned} \int \cos(x+1)\,dx &= \int \cos u\,du && \text{Substituting in,} \\ &= \sin u + c && \text{Integrating with respect to } u, \\ &= \sin(x+1) + c && \text{Substituting back so our answer is in terms of } x \end{aligned}$$

And that's it! ✠

These examples have been nice because $du = dx$, but that is not always true. But it's not too much harder:

Ex 3: *Find* $\int 2x\cos(x^2)\,dx$.

Solution: We look inside parentheses to find u, so

$$\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x && \text{Differentiating,} \\ du &= 2x\,dx \end{aligned}$$

And now we can do our substituting:

$$\begin{aligned} \int 2x\cos(x^2)\,dx &= \int \cos(x^2)2x\,dx \\ &= \int \cos u\,du \\ &= \sin u + c \\ &= \sin(x^2) + c \end{aligned}$$

✠

And, sometimes, things are even more complicated with the dx bit. Look at the differences between the previous example and the next example. The only difference is a 2,

Ex 4: *Find* $\int x\cos(x^2)\,dx$.

Solution: Inside the parentheses, $u = x^2$, but looking at the integral, we can substitute $u = x^2$ into $\cos(x^2)$, getting $\cos u$, and then what's left over, zinazobaki, is $x\,dx$. However,

$$\begin{aligned} \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ du/2 &= x\,dx \end{aligned}$$

Which is what we need to proceed.

$$\begin{aligned}\int x\cos(x^2)\,dx &= \int \cos(x^2)x\,dx\\ &= \int \cos(u)\,\frac{du}{2}\\ &= \frac{1}{2}\int \cos u\,dx\\ &= \frac{1}{2}\sin u + c\\ &= \frac{1}{2}\sin(x^2)+c\end{aligned}$$

✠

The most important thing to remember with u-substitution is that you must do it completely. You cannot integrate something that has both u and x, or u and dx, it must be completely u's with a du or completely x's with a dx. No mixing of u's and x's!

Exercises

Evaluate the integrals.

3.4.1:

(**a**) $\int (x+1)^3\,dx$ (**b**) $\int (x+2)^{-3}\,dx$

(**c**) $\int 2x(x^2+4)^2\,dx$ (**d**) $\int 2x\cos(x^2)\,dx$

3.4.2:

(**a**) $\int \sec^2(x+\pi)\,dx$ (**b**) $\int \frac{9x^2+4x}{3x^3+2x^2+1}\,dx$

(**c**) $\int 4e^{x+2}\,dx$ (**d**) $\int \sin x\cos x\,dx$

3.4.3:

(**a**) $\int xe^{x^2}\,dx$ (**b**) $\int (6x^2-4)(x^3-2x)^{-1}\,dx$

(**c**) $\int x\sin(\omega x^2+\phi)\,dx$ (**d**) $\int x(3x^2-4)^3\,dx$

3.4.4: (NECTA 2008) Integrate the following expression with respect to x: $\frac{\ln x}{x}$. (1 mark)

3.4.5: (NECTA 2008) Evaluate

$$\int_0^1 \frac{x}{\sqrt{3x^2+1}}\,dx$$

(4 marks)

3.4.6: (NECTA 2005) Evaluate

$$\int_0^1 \frac{4x\,dx}{(2-x^2)^{3/2}}$$

(3 marks)

3.4.7: (NECTA 2003) Evaluate

$$\int_0^1 (3+e^x)(2+e^{-x})\,dx$$

(2 marks)

3.4.8: (NECTA 2003) By using a suitable substitution, find

$$\int \frac{1}{\sqrt{(x-3)}}\,dx$$

(2 marks)

3.4.9: (NECTA 2002) Evaluate

$$\int_0^2 \frac{x}{(x^2+1)^2}\,dx$$

(3 marks)

3.4.10: (NECTA 2000)

$$\int \sin x \cos x\,dx$$

(2 marks)

3.4.11: (NECTA 2000) Find

$$\int 5x^4 e^{x^5}\,dx$$

(3 marks)

3.5 Integration by Parts

Integration by Parts is equivalent to the Product Rule from differentiation. The Product Rule states that

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u).$$

If we integrate both sides, we can get the following formula:

$$\begin{aligned}
\int \frac{d}{dx}(uv)\,dx &= \int u\frac{d}{dx}(v)\,dx + \int v\frac{d}{dx}(u)\,dx \\
uv &= \int u\frac{d}{dx}(v)\,dx + \int v\frac{d}{dx}(u)\,dx \\
\int u\frac{d}{dx}(v)\,dx &= uv - \int v\frac{d}{dx}(u)\,dx \\
\int u\,dv &= uv - \int v\,du
\end{aligned}$$

This last line is the formula we use. To be helpful, you need to pick u and dv out of your original integral in a way so that $\int v\,du$ is easier than the original. Here it is one more time:

$$\int u\,dv = uv - \int v\,du$$

Ex 1:

$$\int xe^x \, dx$$

Solution: We need to find a u and a dv in our equation, $xe^x \, dx$. We want to find them so that $v \, du$ will be easy to integrate. It looks like $u = x$ is a very good choice because then $du = dx$ so $v \, du$ is simple. So, we start like this:

$$\begin{aligned} u &= x & v &=? \\ du &=? & dv &= e^x dx \end{aligned}$$

Now, we fill in the ?'s. $u = x$ so $\frac{du}{dx} = 1$, which means that $du = dx$. $dv = e^x dx$, we integrate to find $v = \int dv = \int e^x dx = e^x$. Now we can complete our substitution chart:

$$\begin{aligned} u &= x & v &= e^x \\ du &= dx & dv &= e^x dx \end{aligned}$$

Substituting in and integrating:

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ \int x \, e^x dx &= xe^x - \int e^x \, dx \\ \int x \, e^x dx &= xe^x - e^x + c \end{aligned}$$

And that's that. ✠

In choosing u and dv remember that you must be able to differentiate u and integrate dv.

Ex 2:

$$\int x \ln x \, dx$$

Solution: A first guess might be to let $u = x$ and $dv = \ln x \, dx$. But, then you remember that we can't integrate $\ln x \, dx$! Haiwezakani! So that can't possibly work as dv. So, switch it around.

Let $u = \ln x$ and $dv = x \, dx$. Then $du = 1/x \, dx$ and $v = x^2/2$.

$$\begin{aligned} u &= \ln x & v &= x^2/2 \\ du &= \frac{1}{x} dx & dv &= x \, dx \end{aligned}$$

Substituting in and integrating:

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ \int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx \\ &= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c \end{aligned}$$

✠

When integrating by parts, just like in U-Substitution, you must substitute completely. Every part of the original integral needs to be either in u or in dv. It's a long process, but it is the only way to integrate some functions.

Ex 3:

$$\int_0^1 \frac{x}{\sqrt{x+1}}$$

Solution: Definite integration by parts is no different. We will use $u = x$, because it's derivative is easy. That leaves $dv = dx/\sqrt{x+1} = (x+1)^{-1/2}dx$.

$$u = x \qquad v = 2(x+1)^{1/2}$$
$$du = dx \qquad dv = (x+1)^{-1/2}dx$$

Substituting and integrating, carrying the bounds until the end:

$$\begin{aligned}
\int_0^1 u\,dv &= \Big[uv\Big]_0^1 - \int_0^1 v\,du \\
\int_0^1 \frac{x}{\sqrt{x+1}} &= \Big[2x(x+1)^{\frac{1}{2}}\Big]_0^1 - \int_0^1 2(x+1)^{\frac{1}{2}}\,dx \\
&= \Big[2x(x+1)^{\frac{1}{2}}\Big]_0^1 - \Big[2\cdot\frac{2}{3}\cdot(x+1)^{\frac{3}{2}}\Big]_0^1 \\
&= \Big[2\cdot 1(1+1)^{\frac{1}{2}} - 0\Big] - \Big[\frac{4}{3}(1+1)^{\frac{3}{2}} - \frac{4}{3}(0+1)^{\frac{3}{2}}\Big] \\
&= 2\sqrt{2} - \frac{4\sqrt{8}}{3} + \frac{4}{3} \\
&= \frac{6\sqrt{2}}{3} - \frac{8\sqrt{2}}{3} + \frac{4}{3} \\
&= \frac{4-2\sqrt{2}}{3}
\end{aligned}$$

✠

Exercises

If you have too much trouble one way, try a different u and dv.

3.5.1: (NECTA 2003) By using a suitable substitution, find (2 marks)

$$\int \frac{x}{\sqrt{(x-3)}}\,dx$$

Note: By saying 'suitable substitution' you may think it is possible using u-substitution. Kumbe, it is not possible with u-substitution. You must integrate by parts.

3.6 Volume by Revolution

If you take a plane figure (like a rectangle, triangle, or half-circle), and revolve it around one of its edges—or even another line in that plane—you can generate a 3-dimensional figure. For example, a half-circle, rotated around its straight edge, will give a sphere, or a rectangle rotated about one of its edges will give a cylinder. If you take a notebook, and spin it around its edge, you can maybe see the cylinder. Even try leaving some of the pages open, and they will be like an 'outline' of the cylinder.

These solids are a little difficult to imagine, but it is very easy to find their volumes using integration. it is analogous to finding area with definite integrals. In finding area, we have infinitely many small rectangles of height $f(x)$ and width dx, and then, when we integrate $\int_a^b f(x)\,dx$, we get the sum of all these little areas, the area under the curve.

To find volume, we look at little disks of radius $f(x)$. The disks have this radius, and width dx, so to calculate their volume, it is $\pi[f(x)]^2\,dx$, and when we integrate the volume of all these little disks, $\int_a^b \pi[f(x)]^2\,dx$, we get the sum, which is the volume of the revolution.

Definition: *If a solid is obtained by revolving the area under $f(x)$ from $x = a$ to $x = b$ about the x-axis, the volume of the resulting solid is given by*

$$V = \int_a^b \pi\big[f(x)\big]^2\,dx.$$

Ex 1: *The area between the x-axis and the curve $y = \sqrt{x}$ for $4 \le x \le 9$ is revolved around the x-axis to obtain a solid. Find the volume.*
Solution: Using our formula,

$$\begin{aligned} V &= \int_4^9 \pi\big[\sqrt{x}\big]^2\,dx \\ &= \pi\int_4^9 x\,dx \\ &= \pi\left[\frac{x^2}{2}\right]_4^9 \\ &= \pi[81/2 - 16/2] \\ &= \frac{65\pi}{2} \end{aligned}$$

This can also be used to find some general formulas. For example, a right circular cone:

Ex 2: *A right circular cone is generated by revolving the area under $y = x$ from $x = 0$ to $x = 1$ about the x-axis.* **(a)** *Find the volume of this cone.* **(b)** *Find the volume for a general cone of height h and radius r.*
Solution: **(a)** This one is not so tough, we just use the formula again.

$$\begin{aligned} V &= \int_0^1 \pi\big[x\big]^2\,dx \\ &= \pi\int_0^1 x^2\,dx \\ &= \pi\left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{\pi}{3} \end{aligned}$$

(b) This one requires a little thought first. A right circular cone has a straight edge, so we want to revolve a straight line, and its good that it goes through the origin, that is the tip of the cone. But what is the equation for the line? We need to picture that x

corresponds to the height of the cone, because the x-axis runs down the center of the solid of revolution. So we are concerned with the interval from $x = 0$ to $x = h$. Now, what about that radius. When $x = 0$ the radius is 0, that's the point of the coin. Therefore, at the other end, when $x = h$ we want the radius to be its full value $f(h) = r$. So we want to choose a slope m such that $f(x) = mx$ and $f(h) = r$. Now we can see that $f(x) = \frac{r}{h}x$, and when $x = h$, $f(h) = r$. So that is our function. To find volume, we use the formula:

$$\begin{aligned} V &= \int_0^h \pi[\frac{r}{h}x]^2\,dx \\ &= \pi\frac{r^2}{h^2}\int_0^h x^2\,dx \\ &= \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \pi\frac{r^2}{h^2}\cdot\frac{h^3}{3} \\ &= \frac{1}{3}\pi r^2 h, \end{aligned}$$

Which, if you look in a geometry book, is indeed the volume of a cone. It is very difficult to find this formula through geometry. Archimedes was the first to do it, people had been trying for almost 1,000 years before him. But he didn't have calculus. With the technique of volume by revolution, we can do it in half a page! ✠

Exercises

3.6.1: (NECTA 2004, 2008) Find the volume of the solid formed when the area between the x-axis, the lines $x = 2$ and $x = 4$, and the curve $y = x^2$ is rotated once about the x-axis. Leave your answer in terms of π. (5 marks in 2008, 7 marks in 2004)

3.7 Applications of Integration

Exercises

3.7.1: (NECTA 2003) Find the mean of $y = \sin x$ over the interval from 0 to π. (2 marks)

3.8 Chapter Revision and Exercises

$$\int k\,du = ku + c$$
$$\int f(u) \pm g(u)\,du = \int f(u)\,du \pm \int g(u)\,du$$
$$\int kf(u)\,du = k \int f(u)\,du$$
$$\int u^n\,du = \frac{1}{n+1}u^{n+1} + c \quad \text{for } n \neq -1$$
$$\int e^u\,du = e^u + c$$
$$\int \frac{1}{u}\,du = \ln u + c$$
$$\int \cos u\,du = \sin u + c$$
$$\int \sin u\,du = -\cos u + c$$
$$\int \sec^2 u\,du = \tan u + c$$

Exercises

3.8.1: (NECTA 2002) Find an expression for the area under the curve $y = x^2(2x^3+3)^5$. (*Note:* This is silly without knowing the range of x values. It would be best to do it in general, from $x = a$ to $x = b$.) (4 marks)

Chapter 4

Vectors and Matrices

Vectors are how we deal with physical quantities in more than 1-dimension. We will look at only 2-dimensional and 3-dimensional vectors, but everything we do can be scaled up to as many dimensions as you want!

4.1 Basic Vector Operations

4.1.1 What is a vector?

Definition: Vectors *are not just numbers. Vectors have 2 parts:* ***magnitude*** *and* ***direction****.*

Regular numbers do not have direction (except for positive and negative).
• Note $\mathcal{O}$n $\mathcal{N}$otation •
There are many different ways to indicate that a variable, 'a', is a vector. All of the following are used by different people:

$$\mathbf{a}, \quad \vec{a}, \quad \underline{a}, \quad \hat{a}$$

and there are more. It is best to choose one. In this book, we will use $\vec{a}$.

Ex 1: *Identify the following as vector or scalar:* ***(a)*** *3kg,* ***(b)*** *2m up,* ***(c)*** *9eV/m^3,* ***(d)*** *East,* ***(e)*** *30m/s South.*
Solution: **(a)** 3kg is a scalar.
(b) 2m up is a vector.
(c) 9eV/m^3 is a scalar.
(d) East is just a direction, neither vector nor scalar.
(e) 30m/s South is a vector. ✠

Vectors, as we said, have magnitude and direction. They do not have a specific starting place. Thus 2 vectors are equal if their magnitudes are equal and their directions are equal, even if they are in different places.

We often write vectors in terms of *unit vectors*. Just like the unit circle is a circle of radius 1, a unit vector is a vector of magnitude 1. We use a 'hat' $\hat{}$ instead of an arrow $\vec{}$ to denote a unit vector. The 3 main unit vectors are $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. Usually we think of $\hat{\imath}$ as being in the x-direction, $\hat{\jmath}$ in the y-direction, and $\hat{k}$ perpendicular to them both, in the z-direction.

If you are looking at a paper, usually we think of x and y as being just like normal 2-dimensions, but if there is z then imagine the z-axis as coming straight up out of the paper towards your face. That is the way to imagine it, but when we draw it, we must do it like in the diagram.

There are two good ways to write a specific vector. If a certain vector is 2 units in the positive x-direction, 3 units in the $-y$-direction, and 4 units in the positive z direction, we can

3-D Coordinate System.

Figure 4.1: Drawing in 3 Dimensions.

write either

$$2\hat{\imath} - 3\hat{\jmath} + 4\hat{k} \qquad \text{or} \qquad \begin{pmatrix} 2 \\ -3v \\ 4 \end{pmatrix}.$$

These mean the same thing. Sometimes we want to write vectors just as the definition states them, as a magnitude and a direction. This is most common in two dimensions. For example you might have a vector 5 units at a 45° angle. As usual, we measure angles from the x-axis.

Ex 2: *The line from* $(1, 2)$ *to* $(5, 5)$ *is a vector, call it* $\vec{v}$. *Find its magnitude,* $|\vec{v}|$, *its direction, and write it in* $\hat{\imath}, \hat{\jmath}$ *form.*

Solution: The horizontal distance between the points is $5 - 1 = 4$, and the vertical distance between the points is $5 - 2 = 3$. Because the vector starts at $(1, 2)$ and goes to $(5, 5)$, it is positive in both horizontal and vertical directions. Thus we can write it

$$4\hat{\imath} + 3\hat{\jmath}.$$

Finding its magnitude is just using the distance formula or Pythagorean Theorem, like way back in Section 1.4. So

$$|\vec{v}| = \sqrt{4^2 + 3^2} = 5.$$

To find its direction, we use trigonometry. The angle θ it makes with the horizontal is given by $\tan\theta = 3/4$, which yields

$$\theta = 36.87°.$$

✠

Every 2-dimensional vector can be expressed as $\vec{u} = a\hat{\imath} + b\hat{\jmath}$, and its magnitude will be $|\vec{u}| = \sqrt{a^2 + b^2}$.
Every 3-dimensional vector can be expressed as $\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, and its magnitude will be $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$.

If a vector is given as a magnitude and a direction, you can use sine and cosine to find the $\hat{\imath}$, $\hat{\jmath}$ (and $\hat{k}$) components.

4.1.2 Addition and Subtraction of Vectors

Vector addition and subtraction is intuitive: it works just like you want it to. If $\vec{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ and $\vec{v} = d\hat{\imath} + e\hat{\jmath} + f\hat{k}$, then addition is defined as:

$$\vec{u} + \vec{v} = (a + d)\hat{\imath} + (b + e)\hat{\jmath} + (c + f)\hat{k}$$

Which is to say each component adds and nothing strange happens. In alternative notation:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a + d \\ b + e \\ c + f \end{pmatrix}$$

Subtraction is just the same, but with $-$ instead of $+$. Vector addition and subtraction is only defined if the vectors have the same number of dimensions. If they are different sizes, it is undefined.

$$\begin{pmatrix} 3 \\ 1 \\ -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \quad \text{Undefined}$$

Ex 3: *Find **(a)** $\vec{a}+\vec{b}$ for $\vec{a} = 2\hat{\imath} - 3\hat{\jmath}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath}$, and*

***(b)** $\vec{p}+\vec{q}$ for $\vec{p} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 0 \\ -4 \\ 5 \end{pmatrix}$.*

Solution: (a)

$$\begin{aligned} \vec{a}+\vec{b} &= 2\hat{\imath} - 3\hat{\jmath} + 5\hat{\imath} - \hat{\jmath} \\ &= 7\hat{\imath} - 4\hat{\jmath}. \end{aligned}$$

(b)

$$\vec{p}+\vec{q} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

✠

What we have just seen is the *algebraic* meaning of vector addition. But it is also important to understand vector addition *geometrically*.

4.1.3 Scalar Multiplication of Vectors

Just like addition and subtraction, scalar multiplication of vectors is very easy algebraically (also intuitive), but also must be understood geometrically. But first, a warning:

`Warning!` Never say just 'vector multiplication' or 'a times b' if $\vec{a}$ and $vecb$ are vectors. There are 3 kinds of vector multiplication, and they are very different. You must say 'dot' if it is a dot product, or 'cross' if it is a cross product. The first kind, scalar multiplication, is what we mean usually by 'times'. A scalar times a vector. If $\vec{v} = a\hat{\imath} + b\hat{\jmath} = c\hat{k}$ and k is a scalar, then scalar multiplication is defined as:

$$\vec{v} = (ka)\hat{\imath} + (kb)\hat{\jmath} + (kc)\hat{k}$$

Which is to say each component gets multiplied and nothing strange happens. In alternative notation:

$$k \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$$

Multiplying by a scalar 'scales' a vector. If the scalar is positive, changes the magnitude, but leaves the direction the same. If the scalar is negative is makes the direction opposite and multiplies the magnitude. This is to say:

$$|k \cdot \vec{v}| = |k| \cdot |\vec{v}|$$

The magnitude of a scalar times a vector is the absolute value of the scalar times the magnitude of the vector. This is most useful when you want to find unit vectors. As we said before, a unit vector is any vector of magnitude 1. The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ also have specific directions, but we can find a unit vector pointing in any direction, just by dividing a vector by its magnitude. For any vector $\vec{n}$, the unit vector in the direction of $\vec{n}$ is given by

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}.$$

Ex 4: *Find a unit vector in the direction of* $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

Solution: First we find $|\vec{a}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$. Now, we want to scale $\vec{a}$ down to a unit vector without changing its direction. So we just divide by its magnitude:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{14}}\hat{\imath} - \frac{2}{\sqrt{14}}\hat{\jmath} + \frac{1}{\sqrt{14}}\hat{k}$$

✠

4.1.4 Vector Equation of a Line

We have already said that vectors do not have a set starting or stopping point. In general, this is true, but sometimes we like to use *position vectors*, which usually have a variable (like t, for time), and they give the position of something in terms of that variable (like the position of a particle at time t). So, position vectors are always starting at the origin, and they describe position relative to the origin.

Another common type of vector is a *displacement vector*. Displacement is change in position. A displacement vector starts at one position and goes to another position. For example, if a particle moves from position vector $\vec{p}_1$ to $\vec{p}_2$, the displacement vector, the actual distance traveled, is given by $\vec{d} = \vec{p}_2 - \vec{p}_1$.

We can use this idea to understand the vector equation of a line. A line parallel to vector $\vec{m}$ going through a point with position vector $\vec{a}$ has vector equation

$$\vec{r} = \vec{a} + \lambda\vec{m},$$

where $\vec{r}$ is the position vector for any point on the line, and λ is any real number. The vector equation for a line is a lot like point-slope form. All you need to know is the position vector of one point $\vec{a}$, and the slope in vector form $\vec{m}$. It is also good because it works just the same in 2-dimensions as in 3-dimensions.

Ex 5: *Find the vector equation of a line parallel to* $\vec{m} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ *that goes through the point with position vector* $5\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

Solution: $\vec{r} = 5\hat{\imath} - 2\hat{\jmath} + 4\hat{k} + \lambda(2\hat{\imath} - \hat{\jmath} + 3\hat{k})$ is the vector equation of the line. ✠

We can take the vector equation in the last example and write it out using $\hat{\imath}$ as x, $\hat{\jmath}$ as y, and $\hat{k}$ as z, and find a *parametric equation* of the line, like this:

$$\begin{aligned} x &= 5 + 2\lambda \\ y &= -2 - \lambda \qquad \text{Parametric Equations for a Line in 3-D} \\ z &= 4 + 3\lambda \end{aligned}$$

In 2-dimensions we can convert from vector equation to parametric equation to point-slope or slope-intercept form. It goes like this:

Ex 6: *Find equation in* ***(a)*** *vector form,* ***(b)*** *parametric form, and* ***(c)*** *slope-intercept form for a line passing through* $(-1, 2)$ *and parallel to the vector* $\vec{m} = 3\hat{\imath} - 4\hat{\jmath}$.

Solution: **(a)** Vector form is nice and easy: $\vec{r} = -\hat{\imath} + 2\hat{\jmath} + \lambda(3\hat{\imath} - 4\hat{\jmath})$.

(b) After we have vector form, we just need to separate the $\hat{\imath}$ and $\hat{\jmath}$ components to find parametric form.

$$\begin{cases} x = -1 + 3\lambda \\ y = 2 - 4\lambda \end{cases}$$

(c) We have two options, we can calculate slope as $\frac{\Delta y}{\Delta x} = \frac{-4}{3}$ and go straight to point slope form: $y - 2 = -4/3(x+1)$ and then make y the subject to find slope-intercept form. Or, we can use the parametric form, and find that

$$\begin{aligned}
x &= -1 + 3\lambda && \text{Making } \lambda \text{ the subject,}\\
\lambda &= \frac{x+1}{3} && \text{Which we can then substitute}\\
y &= 2 - 4\lambda && \text{Into the equation for } y\text{...}\\
&= 2 - 4\frac{x+1}{3} &&\\
y &= \frac{-4}{3}x + \frac{2}{3} &&
\end{aligned}$$

✠

Exercises

4.1.1: (NECTA 2006) Given the points $A(2,-1)$ and $B(-3,3)$, find:
(a) A vector from point A to point B in terms of unit vectors $\hat{\imath}$ and $\hat{\jmath}$.
(b) The length of vector $\overrightarrow{AB}$.
(c) The unit vector in the direction of vector $\overrightarrow{BA}$. (5 marks)

4.1.2: (NECTA 2005) If $\vec{a} = 4\hat{\imath} - 3\hat{\jmath}$, $\vec{b} = 2\hat{\imath} + 4\hat{\jmath}$, and $\vec{c} = 22\hat{\imath} - 11\hat{\jmath}$, find the value of scalars m and n for which $m\vec{a} + n\vec{b} = \vec{c}$. (4 marks)

4.1.3: (NECTA 2003) Given $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$, find $\vec{a} + \vec{b}$. (1 mark)

4.1.4: (NECTA 2003) If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath}$ are position vectors of A and B, respectively, find the position vector $\vec{c}$ of C, which divides AB internally in the ration 1:2. (*Note:* This problem is unclear about the ratio. I assume it means to find the position vector of point C such that the ratio of lengths $\overline{AC} : \overline{BC} = 1 : 2$.) (4 marks)

4.2 The ·Dot· Product

`Warning!` Never say just 'vector multiplication.' There are 3 kinds of vector multiplication, and they are very different. The first kind, scalar multiplication, which is called 'times,' was covered in the previous section. This section is about the second kind, the Dot Product. When you have two vectors with a dot product, $\vec{a} \cdot \vec{b}$, you must say 'a DOT b', use 'times' for scalar multiplication.

Definition: *The* dot product *of two vectors, written $\vec{a} \cdot \vec{b}$ and read 'a dot b,' is defined as*

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta,$$

where θ is the angle between them. Usually the simplest way to find the dot product is to use the following formula:

$$\begin{aligned}
\text{If } \vec{v}_1 &= a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k},\\
\text{And } \vec{v}_2 &= a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k},\\
\text{Then } \vec{v}_1 \cdot \vec{v}_2 &= a_1a_2 + b_1b_2 + c_1c_2.
\end{aligned}$$

Or, to use the alternative notation:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = a_1a_2 + b_1b_2 + c_1c_2.$$

Notice that **the answer is a scalar**. Dot product takes two vectors input, and the output is a scalar.

What the dot product does is give you a number that is bigger when the vectors are bigger, and when they are in the same direction. In fact, the dot product is in the true definition of **work** in physics.

Definition: Work *is defined as* $\vec{F} \cdot \vec{d}$ *where* $\vec{F}$ *is the force applied and* $\vec{d}$ *is the distance moved.*

There are two ways to calculate the dot product, so it is often very useful for finding the angle θ between two vectors. Because $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$, if the vectors are perpendicular, then $\theta = \pi/2 = 90°$, so $\cos\theta = 0$. Thus

$$\vec{a} \cdot \vec{b} = 0 \text{ if and only if } \vec{a} \perp \vec{b}.$$

Also useful, because $\vec{a} \cdot \vec{b} = |a||b| \cos\theta$ we can make $\cos\theta$ the subject and find that

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}\,.$$

Ex 1: *For* $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$ *and* $\vec{b} = \hat{\imath} - 3\hat{\jmath} + \hat{k}$ *find* ***(a)*** $\vec{a} \cdot \vec{b}$, *and* ***(b)*** *the value of the angle between them.*

Solution: (a) $\vec{a} \cdot \vec{b} = 2 \cdot 1 + (-3)(-3) + 5 \cdot 1 = 16.$

(b)

$$\begin{aligned} |\vec{a}| &= \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}, \\ |\vec{b}| &= \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{11} && \text{Thus, we know that: } \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} \\ \cos\theta &= \frac{16}{\sqrt{11} \cdot \sqrt{38}} && \text{We can now use a calculator to find } \theta, \\ \theta &= 38.5°. \end{aligned}$$

✠

Ex 2: *Find a unit vector perpendicular to* $\vec{u} = 4\hat{\imath} - \hat{\jmath}$.

Solution: Let $\vec{v} = x\hat{\imath} + y\hat{\jmath}$. We want $\vec{v}$ to be perpendicular to $\vec{u} = 4\hat{\imath} - \hat{\jmath}$, so that means $\vec{u} \cdot \vec{v} = 0$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 4 - y = 0 \\ 4x &= y \end{aligned}$$

Any solution to this equation will be perpendicular to $\vec{u}$. There are infinitely many solutions, just like there are infinitely many vectors perpendicular to $\vec{u}$. So, let's pick one, like $x = 1$ and $y = 4$, thus $\vec{v} = \hat{\imath} + 4\hat{\jmath}$. This is a vector perpendicular to $\vec{u}$, but we need a *unit* vector, so we just divide by magnitude:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{\imath} + 4\hat{\jmath}}{\sqrt{1^2 + 4^2}} = \frac{1}{\sqrt{17}}\hat{\imath} + \frac{4}{\sqrt{17}}\hat{\jmath}$$

is a unit vector perpendicular to $\vec{u}$. ✠

The Zero Vector, consisting of all 0's $(0\hat{\imath}+0\hat{\jmath}+0\hat{k})$, is often written as $\vec{0}$. It has no magnitude, but *every* direction. It is perpendicular to every other vector and parallel to every other vector. It is 0 units in any direction. Generally, it is an exception to rules, and we will try not to use if very much. In the example above, we could have picked $\vec{0}$ as a vector perpendicular to $\vec{u}$, but then we would have been defeated when we tried to turn it into a unit vector. It is best to avoid the zero vector.

The dot product also is nice in that it distributes over addition, which means that:

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

However, scalar multiplication expands and factors like normal, i.e.

$$k(\vec{a}\cdot\vec{b}) = (k\vec{a})\cdot\vec{b} = \vec{a}\cdot(k\vec{b})$$

The proofs of these are easy. You can do them if you try! Another nifty dot product fact is that $\vec{a}\cdot\vec{a} = |\vec{a}|^2$, since the angle between $\vec{a}$ and itself is $0°$.

4.2.1 Projections

The dot product is also useful for finding the *projection* of one vector on another. A projection is like a shadow, it shows you how much a certain vector goes in the direction of a different vector. If you hold a ruler up at an angle from the desk, the shadow of the ruler on the desk shows you how far the ruler goes in the direction of the desk.

Definition: *The projection of a vector $\vec{b}$ onto another vector $\vec{a}$ is written:*

$$\mathrm{proj}_{\vec{a}}\,\vec{b}.$$

The magnitude of the projection we can find from simple trigonometry: it will be $|\vec{b}|\cos\theta$. But

$\boxed{\text{Projection Diagram}}$

Figure 4.2: The projection of $\vec{b}$ onto $\vec{a}$.

we know that

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

so we can say that

$$\left|\mathrm{proj}_{\vec{a}}\,\vec{b}\right| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}.$$

And what is the direction of the projection? Of course it is the same direction as $\vec{a}$, so if we just multiply its magnitude (above) by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$, we will have the projection.

$$\mathrm{proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

And, do remember that $\vec{a}\cdot\vec{a} = |\vec{a}|^2$.

Ex 3: *Find the projection of $\vec{v} = 3\hat{\imath}-4\hat{\jmath}+2\hat{k}$ onto $\vec{u} = -\hat{\imath}+3\hat{\jmath}-5\hat{k}$.*

Solution: We just use the formula. That's all there is to it.

$$\begin{aligned}
\operatorname{proj}_{\vec{u}} \vec{v} &= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u} \\
&= \frac{(-3)+(-12)+(-10)}{1+9+25}\,(-\hat{\imath}+3\hat{\jmath}-5\hat{k}) \\
&= \frac{-25}{35}\,(-\hat{\imath}+3\hat{\jmath}-5\hat{k}) \\
&= \frac{5}{7}\hat{\imath}-\frac{15}{7}\hat{\jmath}+\frac{25}{7}\hat{k}
\end{aligned}$$

✠

Exercises

4.2.1: (NECTA 2008) If A, B, and C are points $(-1,3,-1)$, $(3,5,-5)$, and $(2,-2,1)$, respectively, find the cosine of the angle θ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (4 marks)

4.2.2: (NECTA 2005) Find the value of x which makes $\begin{pmatrix} x \\ 2 \\ 3 \end{pmatrix}$ perpendicular to $\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$. (2 marks)

4.2.3: (NECTA 2005) Determine the value of λ so that $\vec{a} = 2\hat{\imath}_\lambda\hat{\jmath} + \hat{k}$ and $\vec{b} = 4\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ are perpendicular. (2 marks)

4.2.4: (NECTA 2005) For any two non-zero vectors $\vec{a}$ and $\vec{b}$, is $\vec{a}-\vec{b}$ is perpendicular to $\vec{a}+\vec{b}$, show that $\vec{a} = \vec{b}$. (4 marks)

4.2.5: (NECTA 2003) For the vectors $\vec{a} = \hat{\imath} + \hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath}$, find
(a) the acute angle θ between $\vec{a}$ and $\vec{b}$.
(b) the resolved part of $\vec{a}$ in the direction of $\vec{b}$. (6 marks)

4.2.6: (NECTA 2001) Find the angle between the lines

$$y = x\sqrt{3} + 2 \qquad \text{and} \qquad y\sqrt{3} = x - 4.$$

(2 marks)

4.2.7: (NECTA 2001) Given that $\vec{r} = 2\hat{\imath} = 3\hat{\jmath}$, find the length of vector $\vec{r}$ and the angle it makes with $\vec{i}$. (3 marks)

4.2.8: (NECTA 2001) Find the projection of vector $\vec{a}$ on vector $\vec{b}$ given that $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ and $\vec{b} = 4\hat{\imath} - \hat{\jmath} + \hat{k}$. (3 marks)

4.2.9: (NECTA 2000) Find to the nearest degree the angle between vectors $\vec{P} = 8\hat{\imath}+\hat{\jmath}+3\hat{k}$m and $\vec{q} = 2\hat{\imath} + 8\hat{\jmath} - 3\hat{k}$. (2 marks)

4.2.10: (NECTA 2000) Find the value of the scalar t if the vectors $t\hat{\imath} = 4\hat{\jmath}+3\hat{k}$ and $3\hat{\imath}+5\hat{\jmath}+t\hat{k}$ are orthogonal. (2 marks)

4.3 The ×Cross× Product

`Warning!` Never say just 'vector multiplication,' or '$\vec{u}$ times $\vec{v}$.' There are 3 kinds of vector multiplication, and they are very different. Scalar multiplication of vectors and the dot product were covered in previous sections. This section is about the last kind, the Cross Product.

Cross Product is also sometimes called 'vector product' because it is the only way to multiply two vectors and have your result be a vector. We write

$$\vec{u} \times \vec{v} = \vec{w},$$

and it is defined so that

$$|\vec{w}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. The direction of $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, according to the Right-Hand Rule. The Right-Hand Rule states that if you use your right hand, and point your first finger in the direction of $\vec{u}$, your second finger in the direction of $\vec{v}$, then you point your thumb up, your thumb will be the direction of $\vec{u} \times \vec{v}$. Or you can curl your finger from $\vec{u}$ to $\vec{v}$, and again your thumb will show the direction of $\vec{u} \times \vec{v}$. This means that the **cross**

$\boxed{\text{Right-Hand Rule.}}$

Figure 4.3: Right-Hand Rule

product only works in 3-dimensions. The cross product is not defined for 2-dimensions.
• Note On Notation •
Some books will write $\vec{u} \wedge \vec{v}$ instead of $\vec{u} \times \vec{v}$. They mean the same thing, but we will continue to use $\times$.

Definition: *If $\vec{a} = x_a\hat{\imath} + y_a\hat{\jmath} + z_a\hat{k}$, and $\vec{b} = x_b\hat{\imath} + y_b\hat{\jmath} + z_b\hat{k}$, then the cross product $\vec{a} \times \vec{b}$ is defined as the determinant of the matrix*

$$\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{pmatrix},$$

which is given by

$$(y_a z_b - z_a y_b)\hat{\imath} + (x_a z_b - z_a x_b)\hat{\jmath} + (x_a y_b - y_a x_b)\hat{k}.$$

This is not so easy to memorize, so a good way to remember it is as follows: First, you rewrite the first two columns of the matrix after the matrix:

$$\left|\begin{array}{ccc|cc} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ x_a & y_a & z_a & x_a & y_a \\ x_b & y_b & z_b & x_b & y_b \end{array}\right.,$$

Then you multiply along the diagonals, adding the down-right products ($\searrow$), and subtracting the up-right products ($\nearrow$), getting

$$\hat{\imath}y_a z_b + \hat{\jmath}z_a x_b + \hat{k}x_a y_b - x_b y_a \hat{k} - y - b z_a \hat{\imath} - z_b x_a \hat{\jmath},$$

which can then be simplified as above, yielding

$$(y_a z_b - z_a y_b)\hat{\imath} + (x_a z_b - z_a x_b)\hat{\jmath} + (x_a y_b - y_a x_b)\hat{k}.$$

Ex 1: *Find $\vec{u} \times \vec{v}$ if $\vec{u} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{v} = -\hat{\imath} + 2\hat{\jmath} - \hat{k}$.*
Solution: We write our matrix with the extra columns:

$$\left|\begin{array}{ccc|cc} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ 1 & 2 & -1 & 1 & 2 \\ -1 & 2 & -1 & -1 & 2 \end{array}\right.,$$

And we and products ↘ and subtract products ↗:

$$\begin{aligned}\vec{u}\times\vec{v} &= (-2)\hat{\imath} + (1)\hat{\jmath} + 2\hat{k} - (-2)\hat{k} - (-2)\hat{\imath} - (-1)\hat{\jmath} \\ &= 2\hat{\imath} - 2\hat{\imath} + \hat{\jmath} + \hat{\jmath} + 2\hat{k} + 2\hat{k} \\ &= 2\hat{\jmath} + 4\hat{k}\end{aligned}$$

✠

4.3.1 Properties of the Cross Product

This will just be a quick presentation of some of the important facts about the cross product. A summary can be found at the end of the chapter on page **??**.

$$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a}) \qquad \text{Not commutative. Order matters!}$$

$$\begin{array}{lll} \hat{\imath}\times\hat{\jmath} = \hat{k} & \hat{\jmath}\times\hat{k} = \hat{\imath} & \hat{k}\times\hat{\imath} = \hat{\jmath} \\ \hat{\jmath}\times\hat{\imath} = -\hat{k} & \hat{k}\times\hat{\jmath} = -\hat{\imath} & \hat{\imath}\times\hat{k} = -\hat{\jmath} \end{array}$$

$$\begin{array}{ll} \vec{a}\times\vec{a} = 0 & \text{A vector cross itself is 0} \\ k(\vec{a}\times\vec{b}) = (k\vec{a})\times\vec{b} = \vec{a}\times(k\vec{b}) & \text{Scalars work like normal multiplication.} \\ \vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c} & \text{It distributes over addition.} \\ \vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c} & \text{But it does not associate!} \\ |\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\sin\theta & \text{Where } \theta \text{ is the angle between them.} \end{array}$$

Imagine a parallelogram with sides $\vec{a}$ and $\vec{b}$. The area of a parallelogram is base times height, just like a square. The height of the parallelogram is $\vec{a}\sin\theta$, so the area of the parallelogram is given by $\vec{a}\times\vec{b}$. This is a good way to understand the magnitude of a cross product.

Parallelogram with sides $\vec{a}$ and $\vec{b}$

Figure 4.4: Cross product for area.

You can see that the cross product is the strangest thing you have yet seen. Remember that it does not commute: $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$. Matrix multiplication is also like this, as you will see soon.

Exercises

4.4 Vector Applications

Vectors are extraordinarily useful in many fields, especially computers and physics. All of the pictures of letters and signs in this book are stored on a computer as vectors, so that they can be scaled very easily and they look good whether big or small .

In physics, you learn that many things are vectors, such as position, velocity, force, etc, but you have probably not seen them expressed much in vector form. Not yet. In university physics, most things are done in vector form, because 1-dimension is easy and not at all like real life. A required mathematics course for engineering or advanced chemistry and physics, is Vector Calculus, where you learn to differentiate and integrate vectors. Vector Calculus is useful whenever you have more than just 2 quantities. Good examples would be supply, demand, and time, in economics; concentrations of 2 different chemicals and time in chemistry; and x- $y-$ and z- positions in physics.

Integrating vectors can be difficult, but differentiating is pretty easy, and even shows up on BAM NECTA exams sometimes, so this section will focus on differentiating vectors and some other simple applications.

To start off, we need to remember some basic calculus/physics concepts. We will now use 'vector functions' to represent vector quantities. For example, instead of talking about velocity $v(t)$ at time t, we will talk about the vector velocity $\vec{v}(t)$ at time t.

- If position is given by $\vec{s}(t)$, then velocity $\vec{v}(t) = \vec{s}\,'(t)$ and acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{s}\,''(t)$.
- Momentum, a vector, is defined as $\vec{p} = m\vec{v}$ mass times velocity.
- Kinetic Energy, a scalar, is defined $K = \frac{1}{2}m|\vec{v}|^2$, as half the mass times the magnitude of the velocity squared. And $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Force, a vector, is $\vec{F} = m\vec{a} = \frac{d}{dt}\vec{p}$, mass times acceleration, which is also equal to the derivative of momentum.
- Work, a scalar, is $W = \vec{F} \cdot \vec{d}$, the force vector dot the displacement over which that force acts.
- Power, a scalar, is defined as $P = \vec{F} \cdot \vec{v} = \frac{d}{dt}W$, the dot product of force and velocity, or the derivative of work.
- Angular Momentum, a vector, is defined as $\vec{L} = \vec{r} \times \vec{p}$, the radius cross the linear momentum of the particles.
- Torque (or moment), a vector, is defined as $\vec{\tau} = \vec{r} \times \vec{F} = \frac{d}{dt}(\vec{L})$, where $\vec{r}$ is the displacement vector from the axis of rotation to where the force $\vec{F}$ acts, or the rate of change of Angular Momentum.
- In Electricity and Magnetism, the force $\vec{F}$ on a charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q\vec{v} \times \vec{B}$. Charge q is a scalar, the rest are vectors.

Now, obviously, we need to see what exactly does a vector function look like, and how do you differentiate it? It's pretty easy.

Ex 1: *The position of a particle of at time t is given by $\vec{s}(t) = 3t\hat{\imath} - 2t^2\hat{\jmath} + 5\hat{k}$. If the particle has mass 5 kg, find it's velocity and kinetic energy at time t.*

Solution: We differentiate just like normal.

$$\begin{aligned}\vec{v}(t) &= \vec{s}\,'(t) = 3\hat{\imath} - 4t\hat{\jmath} + 0\hat{k} \\ &= 3\hat{\imath} - 4t\hat{\jmath} \text{ m/s}\end{aligned}$$

From here, kinetic energy is easy, we just need to find the magnitude of velocity and square it.

$$\begin{aligned}K(t) &= m|\vec{v}|^2 \\ &= 5\left(\sqrt{3^2 + (-4t)^2}\,\right)^2 \\ &= 5(9 + 16t^2) \\ &= 45 + 80t^2 \text{ J}\end{aligned}$$

✠

Ex 2: *If a force of 30 N is applied to a box while it is pushed 12m, and the force is applied at a 30° angle to the motion, how much work is done?*

Solution: $W = \vec{F} \cdot \vec{d} = |\vec{F}| \cdot |\vec{d}| \cos\theta$, so we can just multiply $30 \text{ N} \cdot 12 \text{ m} \cdot \cos 30° = 311.8$ J. ✠

Exercises

4.4.1: Methane, CH_4, has geometry with its 4 Hydrogen atoms arranged in a tetrahedron, with the Carbon atom at the center. To get the right shape, we can give the Hydrogen atoms coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$, with the Carbon at $(1/2,\ 1/2,\ 1/2)$. Use dot products of displacement vectors (Carbon to Hydrogen) to show that the bond angle, that is the H-C-H angle, is about 109.5° for any two of the Hydrogen atoms.

4.4.2: A conical pendulum is swinging about the origin in a circle of radius 15cm. It's position at time t is given by the position vector $\vec{r}(t) = 15\cos t\hat{\imath} + 15\sin t\hat{\jmath}$, where t is in seconds. If its mass is 0.25kg, find **(a)** is linear momentum at time t, and **(b)** its angular momentum at time t. Angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, is the cross product of position (displacement from the axis of rotation) and linear momentum.

4.4.3: (NECTA 2006) A particle of unit mass moves so that its position vector $\vec{r}$ at time t seconds is given by

$$\vec{r} = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + \frac{1}{2}t^2\hat{k}.$$

Find the **(a)** Momentum at time t, **(b)**Kinetic energy at time t, **(c)** Force acting on the particle at time t, **(d)** Power exerted by the force in (c) above, at time t.

4.4.4: (NECTA NECTA 2006) At time t, the position vectors of two particles P and Q are given by:

$$\vec{P} = 2t\hat{\imath} + (3t^2 - 4t)\hat{\jmath} - t^3\hat{k}$$
$$\vec{Q} = t^3\hat{\imath} - 2t\hat{\jmath} + (2t^2 - 1)\hat{k}$$

Find the velocity and acceleration of Q relative to P when $t = 3$. (5 marks)

4.5 Basic Matrix Operations

4.5.1 What is a matrix?

A matrix is like a lot of vectors togethers. In fact, a vector is a matrix, with only one column. But matrices (the plural of *matrix* is *matrices*) can be much bigger than vectors. A matrix with r rows and c columns is called an $r \times c$ matrix, read 'r by c.' For example:

$$\begin{pmatrix} 1 & -4 & 2 \\ 8 & 0 & -1 \end{pmatrix} \text{ is a } 2 \times 3 \text{ matrix, and } \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} \text{ is a } 3 \times 1 \text{ matrix.}$$

Matrices are also sometimes written with square brackets [ ] instead of parentheses. They mean the same thing.

$$A = \begin{bmatrix} x & \pi \\ 800 & -6/7 \end{bmatrix} \text{ is a } 2 \times 2 \text{ matrix.}$$

You can put anything you want in a matrix: numbers, fractions, positive, negative, variables, anything is okay. Normally, we call matrices with capital letters, like A, B, C, etc.

Definition: *The* transpose *of a matrix is what you get when you switch its rows and columns.*

If A is a matrix, then A^T, read 'A transpose,' is the matrix with rows equal to A's columns.

Ex 1: *Find the transposes of the following matrices:*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Solution:

$$A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 6 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$$

✠

Notice that if a matrix is $m \times n$, then its transpose will be $n \times n$; the number of rows and columns is switched.

4.5.2 Addition and Subtraction of Matrices

If two matrices are the same size, they can be added or subtracted. Addition and subtraction of matrices is just like addition and subtraction of vectors. Corresponding elements add or subtract, but they do not interact with each other, like this:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \\ g_1+g_2 & h_1+h_2 & i_1+i_2 \end{pmatrix}$$

The matrix size does not matter at all, it is always like this. But, you can only add or subtract matrices that are the same size.

Ex 2: *Find $A+B$, $A-B$, $C+D$, and $B+C$, if*

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ -10 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 & 4 \\ -4 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

Solution:

$$A+B = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -10 & -5 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -8 & -5 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ -10 & -5 \end{pmatrix} = \begin{pmatrix} -2 & -5 \\ 12 & 5 \end{pmatrix}$$

$$C+D = \begin{pmatrix} 0 & 2 & 4 \\ -4 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ -5 & -3 & -1 \end{pmatrix}$$

$B+C$ is undefined because they have different sizes. ✠

If every entry in a matrix is a 0, it is called a 'zero matrix.' These are all zero matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

4.5.3 Scalar Multiplication

In many ways, matrices work like vectors. Addition and subtraction are the same, and so is scalar multiplication. If c is a scalar and A is a matrix, then the product $c \cdot A$ means that you multiply every entry in A by c. For example:

$$\text{If} \quad A = \begin{pmatrix} 0 & 2 & 4 \\ -1 & 2.5 & 10 \end{pmatrix}, \quad \text{then} \quad 3A = \begin{pmatrix} 0 & 6 & 12 \\ -3 & 7.5 & 30 \end{pmatrix}$$

And, just like with vectors (and regular numbers), scalar multiplication *distributes* over matrix addition, which means that

$$k(A + B) = kA + kB,$$

where k is a scalar and A and B are matrices.

4.5.4 Matrix Multiplication

Matrix multiplication is a little more complicated. Remember the vector Dot Product from Section 4.2? Matrix multiplication uses lots and lots of dot products. Also, always remember that when we talk about matrices, we always say 'row by column,' in that order. Row first, column second.

When you multiply two matrices, say $A \cdot B = C$, what you do is take the first *row* from A, and dot it with the first *column* from B, and that is the first entry in C. For the second, you take the first row from A (again), and dot it with the *second* column in B, and that is your second entry. And so on. You always are doing **rows from the first matrix** dotted with **columns from the second matrix**. Just like talking about matrices. Row by column, row by column. Then, the entries go in the row and column of the answer corresponding to the row from the first matrix and the column from the second matrix.

To look at a simple, common example of 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

If you look at the result, C, the entry in the i'th column and the j'th row is the dot product of the i'th row of A and the j'th column of B. If A has r_A rows and c_A columns, and B has r_B rows and c_B columns, then

$$A_{r_A \times c_A} \times B_{r_B \times c_B} = C_{r_A \times c_B}$$

C will have the number of rows of A and the number of columns of B. We'll have a couple small examples, and then some bigger examples.

Ex 3: *Find* ***(a)*** *AB and* ***(b)*** *BA if*

$$A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$$

Solution: (a) A is 3×1 and B is 1×3, so AB will be 3×3.

$$AB = \begin{pmatrix} 3 \cdot -1 & 3 \cdot 0 & 3 \cdot 1 \\ 2 \cdot -1 & 2 \cdot 0 & 2 \cdot 1 \\ 1 \cdot -1 & 1 \cdot 0 & 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 3 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

(b) When we have BA, we see that B is 1×3, A is 3×1, so BA will be 1×1:

$$BA = \begin{pmatrix} -1 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} -2 \end{pmatrix}$$

✠

This last example demonstrates a very important point about matrix multiplication:it **does not commute**. This means that most of the time, even for square matrices,

$$\boxed{AB \neq BA}$$

Ex 4: *Find AB and BA if* $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Solution:

$$AB = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 5 & -1 \end{pmatrix}$$

Once again, $AB \neq BA$. ✠

This is very important when working with equations of matrices. If you are given that

$$PQ = B$$

and you want to multiply through by A, you *must matrix multiply from the same side!*

$PQ = B$	Given. Then multiply through by A,
$A \cdot PQ = A \cdot B$	Either from the left, like this,
or	
$PQ \cdot A = B \cdot A$	Or from the right, like this.
Not: $A \cdot PQ \neq B \cdot A$	You cannot mix sides!

Ex 5: *Find PQ if*

$$P = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix}$$

Solution: Here we see that P is 3×2 and Q is 2×2, so PQ will be 3×2.

$$PQ = \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 & 1 \cdot 8 + 2 \cdot 0 \\ 3 \cdot 7 + 4 \cdot 9 & 3 \cdot 8 + 4 \cdot 0 \\ 5 \cdot 7 + 6 \cdot 9 & 5 \cdot 8 + 6 \cdot 0 \end{pmatrix} = \begin{pmatrix} 25 & 8 \\ 57 & 24 \\ 89 & 40 \end{pmatrix}$$

✠

There is a special square matrix called the *identity*; it has 1's on the diagonal and 0's everywhere else, and we write it I. Multiplying by the identity is like multiplying by 1, $AI = A$ and $IA = A$. To show this:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

There is an identity matrix for every square size, 1×1, 2×2, 3×3 identities are shown.

$$(1)\,, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercises

Use the following for problems **1-21**.

$$A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 6 \\ 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ -1 & -1 \end{pmatrix},$$

$$E = \begin{pmatrix} 2 & 0 \\ 4 & 4 \\ -2 & -2 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Calculate the following, if possible:

4.5.1: **(a)** $A+B$ **(b)** $A+B^T$ **(c)** $C+D$ **(d)** $D+E$

4.5.2: **(a)** $E+C^T$ **(b)** $3D$ **(c)** $2A+B$ **(d)** F^T+F

4.5.3: **(a)** B^T-3F **(b)** $2A+4(B+F^T)$ **(c)** AB **(d)** BA

4.5.4: **(a)** BF **(b)** FB **(c)** DC **(d)** CD

4.5.5: **(a)** AD **(b)** DB **(c)** $(A+B)C$ **(d)** $AC+BC$

4.5.6: Find a scalar x such that $2D + xE = 0$.

4.5.7: Find a matrix M such that $(AB)^T + M = F$.

4.5.8: If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, find

(a) A^2 **(b)** A^3 **(c)** B^2 **(d)** B^3

4.5.9: (NECTA 2008) Find the value of x and y in the following relation:

$$\begin{pmatrix} 3 & -5 \\ 2 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 5 & -2 \end{pmatrix}$$

(2 marks)

4.5.10: (NECTA 2008) If $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix}$, find AB. (2 marks)

4.6 |Determinants| and Inverses^{-1}

What is an inverse? For regular numbers (scalars), x has an inverse x^{-1} such that $x \cdot x^{-1} = 1$. For matrices, it's basically the same. Not all matrices have inverses, but some square matrices have inverses have inverses such that $A \cdot A^{-1} = I$, the identity matrix. To find A^{-1}, we need to know the *determinant* of A, which is written $|A|$. The determinant is not a matrix or vector, it is just a single scalar.

Definition: *For a 2×2 matrix the determinant is defined (and written) as follows:*

$$\textit{If} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \textit{then} \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The inverse of a 2×2 matrix A is then given by:

$$\text{If} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

But, since you divide by the determinant to find the inverse, if the determinant is 0 then there is no inverse. It doesn't exist.

$$\text{If } |A| = 0, \text{ then } A^{-1} \text{ does not exist.}$$

Ex 1: *Find the inverses, if they exist, of the following matrices.*

$$(\mathbf{a})\ A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}, \quad (\mathbf{b})\ B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad (\mathbf{c})\ C = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}.$$

Solution: **(a)** $|A| = 2 \cdot 3 - 4 \cdot (-1) = 10$, so

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3/10 & -2/5 \\ 1/10 & 1/5 \end{pmatrix}.$$

(b) $|B| = 0 \cdot 1 - (-1) \cdot 1 = 1$, so

$$B^{-1} = \frac{1}{1} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) $|C| = 3 \cdot 4 - 2 \cdot 6 = 0$. C does not have an inverse because its determinant is 0. ✠

Way back in Section 1.2 we talked about the inverse of a function. A linear transformation is a function, and it's inverse is the inverse of the linear transformation matrix.

4.6.1 Solving Linear Equations with Matrices

Matrix Algebra is applied everywhere, in computer graphics, solving complicated differential equations, and it is the basis for how the Internet search engine Google works! One easy application of matrices is solving linear equations.

Ex 2: *If you are given the equations*

$$\begin{cases} 3x - 2y = 3 \\ -x + 2y = 4 \end{cases}$$

to solve, you can use matrices.

Solution: Watch how we can rewrite these equations in matrix form:

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 2y \\ -x + y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So what we can do is find the inverse of the big matrix, and then use matrix algebra to solve. Let's call

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

So now our system of linear equations looks like this:

$$AX = M$$

Which means if we find A^{-1}, we can left-multiply both sides, yielding

$$\begin{aligned} A^{-1}AX &= A^{-1}M \\ IX &= A^{-1}M \\ X &= A^{-1}M \end{aligned}$$

So, to find A^{-1} first we need to take the determinant. $|A| = 3 \cdot 2 - (-2) \cdot (-1) = 4$. Thus

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2/4 & 2/4 \\ 1/4 & 3/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

Now we need to multiply $A^{-1}M$ and the result will be X.

$$A^{-1}M = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 + 4/2 \\ 3/4 + 12/4 \end{pmatrix} = \begin{pmatrix} 7/2 \\ 15/4 \end{pmatrix} = X = \begin{pmatrix} x \\ y \end{pmatrix}$$

So $x = 7/2$ and $y = 15/4$. Whenever possible, you should check your answer. For linear equations checking is very easy, so $3x - 2y = 21/2 - 15 - 2 = 6/2 = 3$, the first one is good, and $-x + 2y = -7/2 + 15/2 = 8/2 = 4$ the second checks, we're good! ✠

4.6.2 Linear Transformations

Linear transformations are related to simultaneous linear equations. However, instead of solving them, instead you put in input values and see what comes out. They are just functions.

Definition: *The matrix M is a* linear transformation matrix *that maps the point (x, y) to its* image*, (x', y') by the equation:*

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

This is used for linear equations and functions, like

$$\begin{aligned} x &\mapsto ax + by \\ y &\mapsto cx + dy \end{aligned}$$

In this case the transformation matrix would be

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

because

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{matrix} ax + by \\ cx + dy \end{matrix}$$

Ex 3: *Find the images of the points **(a)** $(1, 0)$ and **(b)** $(3, -4)$ under the linear transformation defined by*

$$M = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

Solution:

$$(\mathbf{a}) \qquad \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The image of $(1, 0)$ is $(2, -3)$.

$$(\mathbf{b}) \qquad \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 10 \\ -25 \end{pmatrix}$$

The image of $(3, -4)$ is $(10, -25)$. ✠

A good linear transformation to know is a rotation. The linear transformation matrix for a rotation by angle θ about the origin is

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We also talked about composition of functions back in Section 1.2. For linear transformations, composition is easy. If you have two matrices M and N, and you want to find M composed with N, $M \circ N$, it's just M times N, MN.

Ex 4: *Let R be the matrix for a rotation of $90°$ around the origin, and let M be the matrix corresponding to the transformation*

$$x \mapsto 2x - 3y$$
$$y \mapsto x + y$$

(a) *Find the image of the point $(2, 1)$ under R.*
(b) *Find the image of your answer in part (a) if M is applied to it.*
(c) *Find the linear transformation matrix for the composite function of first R and then M. Find the image of $(2, 1)$ under this transformation.*
Solution: (a) The rotation matrix R is given by

$$R = \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so the image of $(2, 1)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

which in Cartesian coordinates is $(-1, 2)$.

(b) The matrix M is given by $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$, so applying this to our answer

$$M \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2-6 \\ -1+2 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \end{pmatrix}$$

(c) The composite transformation of first R, then M, is written as

$$MR \begin{pmatrix} x \\ y \end{pmatrix},$$

thus the composite transformation matrix is just MR.

$$MR = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 1 & -1 \end{pmatrix},$$

and the image of $(2, 1)$ is given by

$$\begin{pmatrix} -3 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 1 \end{pmatrix},$$

the same as in part (b). ✠

4.6.3 3×3 Determinants

Finding the determinant of a 3×3 matrix is a little more difficult. The process is just like taking a cross product:

Definition: *The determinant of a 3×3 matrix is given as follows:*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

You can remember this formula in the same way as a cross product, first you rewrite the first two columns

$$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & h & i \end{array}\right|,$$

then you multiply across all the diagonals, adding down-right $\searrow$ and subtracting up-right $\nearrow$, ending up with $\searrow + \searrow + \searrow - \nearrow - \nearrow - \nearrow$.

Ex 5: *Find the determinant of the following matrices:*

$$(\mathbf{a}) \quad P = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 9 \\ 5 & 7 & 6 \end{pmatrix}, \qquad (\mathbf{b}) \quad Q = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 5 & -4 \end{pmatrix}$$

Solution: (a) First we will rewrite P with it first two columns added again at the end:

$$\left|\begin{array}{ccc|cc} 2 & 4 & 0 & 2 & 4 \\ 1 & 3 & 9 & 1 & 3 \\ 5 & 7 & 6 & 5 & 7 \end{array}\right|$$

The we multiply along the diagonals, adding $\searrow$ and subtracting $\nearrow$.

$$\begin{aligned} |P| &= 2 \cdot 3 \cdot 6 + 4 \cdot 9 \cdot 5 + 0 \cdot 1 \cdot 7 - 5 \cdot 3 \cdot 0 - 7 \cdot 9 \cdot 2 - 6 \cdot 1 \cdot 4 \\ &= 36 + 180 + 0 - 0 - 126 - 24 \\ &= 66 \end{aligned}$$

(b) For this one we must be very careful about what is negative and what is positive. Rewrite Q with it first two columns added again at the end:

$$\left|\begin{array}{rrr|rr} 0 & -1 & 2 & 0 & -1 \\ 1 & 1 & 3 & 1 & 1 \\ -2 & 5 & -4 & -2 & 5 \end{array}\right|$$

And then, once again, we multiply along diagonals, adding $\searrow$ and subtracting $\nearrow$.

$$\begin{aligned} |Q| &= 0 \cdot 1 \cdot (-4) + (-1) \cdot 3 \cdot (-2) + 2 \cdot 1 \cdot 5 \\ &\quad - (-2) \cdot 1 \cdot 2 - 5 \cdot 3 \cdot 0 - (-4) \cdot 1 \cdot (-1) \\ &= 0 + 6 + 10 - (-4) - 0 - 4 \\ &= 16 \end{aligned}$$

✠

Finding the inverses of 3×3 matrices is considerably more difficult and will not be covered in this book.

Exercises

4.6.1: (NECTA 2006) Use the inverse matrix method to solve: (5 marks)

$$2y + 3x - 15 = 0$$
$$2x - 20 + 3y = 0$$

4.6.2: (NECTA 2006) If T is a linear transformation such that $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $T(, y) = (3y, 5x)$, find the matrix T, then evaluate $T(0, 0)$. (5 marks)

4.6.3: (NECTA 2005) Solve the following system of equations by using the matrix method:

$$4x + 3y = 31$$
$$9y - x = 41$$

(4 marks)

4.6.4: (NECTA 2003) Given that $A = \begin{pmatrix} 2 & k \\ k & 8 \end{pmatrix}$ is a singular matrix, find the value of k if $k \in \mathbb{R}^+$. (1 mark)

4.6.5: (NECTA 2003) (a) Find the inverse of B given $B = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$.

(b) Use part (a) to solve the following system of system of simultaneous equations:

$$\left.\begin{aligned} 2x + y &= 8 \\ 4x - y &= 10 \end{aligned}\right\}$$

(5 marks)

4.6.6: (NECTA 2003) Solve the following system of equations by using inverse of matrices:

$$x + 2y = 10$$
$$2x - y = 5$$

(3 marks)

4.6.7: (NECTA 2002) A transformation M is given by the matrix M where $M = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
Find:
(a) The image of point $(-2, 5)$ under M.
(b) The inverse of M. (5 marks)

4.6.8: (NECTA 2001) Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ represent a transformation. Find the image of the vector $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ under this transformation. (2 marks)

4.6.9: (NECTA 2001) Find the inverse of the matrix $A = \begin{pmatrix} K & -1 \\ -1 & 0 \end{pmatrix}$. (2 marks)

4.7 Cramer's Rule

Cramer's Rule is a nice way to solve linear equations using matrices, without bothering to find the inverses. All you need to find is the determinants involved. In 2-dimensions, Cramer's Rule states that if you are given

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

which, as we saw in the last section, that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix},$$

then the solution is given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

In words, to find x, the first variable, replace the first column of the matrix with the solution vector. Take the determinant and divide by the determinant of the original matrix, and you get x. To find y, the second variable, replace the *second* column of the matrix with the solution vector. Take the determinant and divide by the determinant of the original matrix, and you get y.

Ex 1: *Use Cramer's Rule to solve*

$$\begin{cases} 3x + 2y = -1 \\ 4x - 3y = 10 \end{cases},$$

Solution: First, we write the equations in vector form:

$$\begin{pmatrix} 3 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix},$$

then, according to Cramer's Rule, the solution is given by

$$x = \frac{\begin{vmatrix} -1 & 2 \\ 10 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{3-20}{-9-8} = 1$$

$$y = \frac{\begin{vmatrix} 3 & -1 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{30 - -4}{-9-8} = -2$$

Thus $x = 1$ and $y = -2$ is our solution. As always, we should check it:

$$3 \cdot 1 + 2 \cdot (-2) = -1 \quad \text{Check!}$$
$$4 \cdot 1 - 3 \cdot (-2) = 10 \quad \text{Check!}$$

So it's good. ✠

Cramer's Rule works just the same in 3-dimensions. A general system of 3 linear equations is:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2\,, \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

which has matrix form
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_1 \\ d_3 \end{pmatrix}$$
To make this simpler, we will let Δ be the main determinant,
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$
and call Δ_n the determinant of this matrix with the solution vector, $\begin{pmatrix} d_1 \\ d_1 \\ d_3 \end{pmatrix}$, substituted into the n^{th} column of the matrix. Then, Cramer's Rule simply states that
$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}.$$
This can even continue on up to 4-dimensions, 5-dimensions, etc. But that is the work of a computer. For us people, we will stop at 3-dimensions.

Ex 2: *Use Cramer's Rule to solve:*
$$\begin{aligned} x - 2y - 3z &= d_1 \\ 3x + 5y + 2z &= d_2\,, \\ 2x + 3y - z &= d_3 \end{aligned}$$

Solution: Writing the equations in matrix form:
$$\begin{pmatrix} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

We can now apply Cramer's Rule.
$$\begin{aligned} \Delta = \begin{vmatrix} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{vmatrix} &= 1\cdot 5\cdot(-1) + (-2)\cdot 2\cdot 2 + (-3)\cdot 3\cdot 3 \\ &\qquad - 2\cdot 5\cdot(-3) - 3\cdot 2\cdot 1 - (-1)\cdot 3\cdot(-2) \\ &= -22 \end{aligned}$$
$$\begin{aligned} \Delta_1 = \begin{vmatrix} 0 & -2 & -3 \\ 0 & 5 & 2 \\ 2 & 3 & -1 \end{vmatrix} &= 0\cdot 5\cdot(-1) + (-2)\cdot 2\cdot 2 + (-3)\cdot 0\cdot 3 \\ &\qquad - 2\cdot 5\cdot(-3) - 3\cdot 2\cdot 0 - (-1)\cdot 0\cdot(-2) \\ &= 22 \end{aligned}$$
$$\begin{aligned} \Delta_2 = \begin{vmatrix} 1 & 0 & -3 \\ 3 & 0 & 2 \\ 2 & 2 & -1 \end{vmatrix} &= 1\cdot 0\cdot(-1) + 0\cdot 2\cdot 2 + (-3)\cdot 3\cdot 2 \\ &\qquad - 2\cdot 0\cdot(-3) - 2\cdot 2\cdot 1 - (-1)\cdot 3\cdot 0 \\ &= -22 \end{aligned}$$
$$\begin{aligned} \Delta_3 = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 5 & 0 \\ 2 & 3 & 2 \end{vmatrix} &= 1\cdot 5\cdot 2 + (-2)\cdot 0\cdot 2 + 0\cdot 3\cdot 3 \\ &\qquad - 2\cdot 5\cdot 0 - 3\cdot 0\cdot 1 - 2\cdot 3\cdot(-2) \\ &= 22 \end{aligned}$$

$$x = \frac{\Delta_1}{\Delta} = \frac{22}{-22} = -1, \qquad y = \frac{\Delta_2}{\Delta} = \frac{-22}{-22} = 1, \qquad z = \frac{\Delta_3}{\Delta} = \frac{22}{-22} = -1,$$

As always, we should check our answers:

$$\begin{aligned} x - 2y - 3z &= -1 - 2 \cdot 1 - 3 \cdot (-1) = 0 \quad \text{Check!} \\ 3x + 5y + 2z &= 3 \cdot (-1) + 5 \cdot 1 + 2 \cdot (-1) = 0 \quad \text{Check!} \\ 2x + 3y - z &= 2 \cdot (-1) + 3 \cdot 1 - (-1) = 2 \quad \text{Check!} \end{aligned}$$

So it's good. ✠

Just like in finding inverses, when we use Cramer's Rule we divide by a determinant. Here, if the determinant $\Delta = 0$, that means that either there is no solution, or there is no unique solution. Obviously, there is no solution to:

$$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases},$$

$x+y$ can be either 1 or 2, but it can't be both. If we try to use Cramer's Rule we will find that

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0,$$

which in this case is telling us that a solution does not exist.

Similarly, there are many solutions to

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}.$$

These equations are not really different. If $x + y = 1$ then it must be true that $2x + 2y = 2$, so the second equation doesn't add any new information. Once again, if we try Cramer's Rule, we will get a 0 determinant:

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0,$$

in this case indicating that there is not a unique solution. Instead, there are infinitely many.

Exercises

4.7.1: (NECTA 2008) Solve the following system of equations by Cramer's Rule.

$$\begin{aligned} x + y + z &= 6 \\ 3x - 2y - z &= -1 \\ 2x + 4y + 3z &= 19 \end{aligned}$$

(6 marks)

4.7.2: (NECTA 2003) Solve the following system of equations:

$$\begin{aligned} x + y + 2 &= 6 \\ 3x - 2y - z &= -1 \\ 2x + 4y + 3z &= 19 \end{aligned}$$

(4 marks)

4.7.3: (NECTA 2002) Solve the following system of equations by the matrix method:

$$\begin{aligned} 2x - 3y + z &= 3 \\ -x + 4y + 3z &= 16 \\ 3x + 2y - 2z &= 1 \end{aligned}$$

(5 marks)

4.7.4: (NECTA 2001) Use Cramer's Rule to solve the following system of simultaneous equations.

$$\begin{aligned} 2x - 2y &= 6 \\ x + 2y &= 0 \end{aligned}$$

(2 marks)

4.8 Chapter Revision and Exercises

Properties of Dot· and Cross× Products

Let $\vec{a} = x_a\hat{\imath} + y_a\hat{\jmath} + z_a\hat{k}$, $\vec{b} = x_b\hat{\imath} + y_b\hat{\jmath} + z_b\hat{k}$, with θ as the angle between $\vec{a}$ and $\vec{b}$, and let $\vec{c}$ also be a 3-dimensional vector. Then

$$\begin{aligned}
&\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = x_a x_b + y_a y_b + z_a z_b \qquad \text{Scalar!}\\
&\vec{a}\times\vec{b} = (y_a z_b - z_a y_b)\hat{\imath} + (z_a x_b - x_a z_b)\hat{\jmath} + (x_a y_b - y_a x_b)\hat{k} \qquad \text{Vector!}\\
&|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta\\
&\vec{a}\cdot\vec{a} = |\vec{a}|^2\\
&\vec{a}\cdot\vec{b} = 0 \quad \text{if and only if} \quad \vec{a}\perp\vec{b}\\
&\vec{a}\times\vec{a} = 0\\
&\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})\\
&\vec{a}\times(\vec{a}\times\vec{b}) = 0 \qquad (\text{because } \vec{a}\times\vec{b}\perp\vec{a})\\
&\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}
\end{aligned}$$

Exercises

4.8.1: (NECTA 2008)
(a) Show that the vectors $3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$ and $4\hat{\imath} + \hat{\jmath} + 8\hat{k}$ are perpendicular.
(b) Given the vectors $\vec{a} = 3\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$ and $\vec{b} = 4\hat{\imath} + \hat{\jmath} + 8\hat{k}$, find the projection of $\vec{a}$ onto $\vec{b}$. (6 marks)

4.8.2: (NECTA 2006) A line passes through the point $(2, -1, 4)$ and is in the direction of vector $\hat{\imath} + \hat{\jmath} - 2\hat{k}$. Find the:
(a) vector equation of the line.
(b) angle the line makes with the positive x axis. (5 marks)

4.8.3: (NECTA 2005) A, B, and C are the points $(-1, 3, -1)$, $(3, 5, -5)$, and $(2, -2, 1)$, respectively. Find the
(a) Distance AB.
(b) Cosine of the angle θ between AB and AC. (4 marks)

4.8.4: (NECTA 2003) If $A = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -5 \\ 9 & 7 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$, show that $A + B - 2C$ is a singular matrix. (3 marks)

4.8.5: (NECTA 2002) Given the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = \hat{\jmath} - \hat{k}$,

(a) Find the vector $\vec{c}$ such that $\vec{a} + 2\vec{b} + \vec{c} = 0$.

(b) What is the sine of the acute angle enclosed by the vectors $\vec{a}$ and $\vec{b}$?

(c) Find the unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$. (10 marks)

4.8.6: (NECTA 2000) If P and Q are the points $(8, 1, 3)$ and $(2, 8, -3)$, respectively, find a unit vector parallel to the displacement vector $\overrightarrow{PQ}$. (2 marks)

4.8.7: (NECTA 2000) Use the matrix inverse method to solve the following simultaneous equations:

$$\begin{aligned} x + 2y &= 8 \\ 4x + 3y &= 22 \end{aligned}$$

(3 marks)

4.8.8: (NECTA 2000) Solve the following system of equations:

$$\begin{aligned} 3x - y + z &= -2 \\ x + 5y + 2z &= 6 \\ 2x + 3y + z &= 0 \end{aligned}$$

(6 marks)

Chapter 5

Probability and Statistics

5.1 Factorial!!! and Permutations

Definition: *A* permutation *is a certain way of arranging some objects. So $n!$ is the number of* permuting*, or ways of ordering, n different objects.*

In our first example, with ABC, we had 3 objects, so we got $3 \cdot 2 \cdot 1 = 6$ possible arrangements. If you look on your calculator you probably have a factorial button that looks like $\boxed{!}$, which will calculate factorials for you.

Ex 1: *How many ways can you arrange the letters of 'shega'?*
Solution: Shega has 5 letters, and they are all different, so our answer is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ✠

This is applicable to more situations than just arranging letters.

Ex 2: *10 girls are running in a race. How many possibilities are there for the 1st, 2nd, and 3rd places?*
Solution: It's not just a factorial, because we are not looking at all 10 places, only the first, second, and third. For first place, there are 10 possibilities, then for second place there are 9 possibilities, and for third place 8 possibilities. Thus the total number of possibilities is $10 \cdot 9 \cdot 8 = 720$. ✠

There's also a better way to do problems like our last example. How many ways can we permute (arrange) r objects if there are n total objects? Just like in the race example, we for the first one we have n choices, for the second $n-1$, so it starts like factorial, but we have to stop after r choices. Thus it is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

For example, in the race, we had

$$10 \cdot (10-1) \cdot (10-2).$$

This is all fine, but it is a little hard to calculate if r is big. But we can write a formula for this in terms of factorial. See how

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) \cdot (n-r) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot (n-r-2) \cdot \ldots \cdot 2 \cdot 1},$$

because everything after $(n-r+1)$ will cancel with the bottom. We can write the same thing using factorial notation:

$$= \frac{n!}{(n-r)!}$$

Definition: *The number of ways to permute r objects from n total objects is called* 'n permute r,' *and is written as*

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Probably there is a button on your calculator to do this also. If not, you can just use factorials.

Exercises

5.1.1: (NECTA 2006) Two dice are thrown together. What is the probability of a score of an 8? (4 marks)

5.1.2: (NECTA 2006) Two of my friends and I play a game of pure chance three times. What is the probability of me winning:
i. Every time? **ii.** Only the third time? (6 marks)

5.1.3: (NECTA 2003) The probability that Juma will be alive in 20 years to come is 0.85, and the probability that his wife will be alive is 0.90. Fid the probability that both will be alive in 20 years to come. (3 marks)

5.1.4: (NECTA 2003) How many different arrangements can be made of the letters in the word "AMEFANIKIWA"? (3 marks)

5.1.5: (NECTA 2003) Over a period of time it is found that 10% of the fuses produced by a certain manufacturing process are defective. Using binomial distribution, find the approximate probability that in a sample of 10 fuses chosen at random, there will be at most 1 which is defective. (*Note:* The binomial distribution has not been covered, but you can still find the answer using other methods.) (4 marks)

5.1.6: (NECTA 2000) Find the number of permutations of the letters of the word YANGA. (2 marks)

5.1.7: (NECTA 2000) Find whether, when a die is thrown, the following pairs of events are mutually exclusive or not.
(a) $\{1, 3, 5\}$, $\{4, 5\}$
(b) $\{1, 2, 3\}$, $\{4, 5, 6\}$ (4 marks)

5.1.8: (NECTA 2008) A bag contains 3 white balls and 2 black balls. Two balls are taken from the bag. What is the probability that one is white and the other is black? (3 marks)

5.1.9: (NECTA 2008) A committee of 5 principals is to be selected from a group of 6 male principals and 8 female principals. If the selection is made randomly, find the probability that there are 3 female principals and 2 male principals. (5 marks)

5.1.10: (NECTA 2002) If $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$, find
(a) $P(A \cup B)$.
(b) $P(A \backslash B)$. (2 marks)

5.1.11: (NECTA 2001) The events A and B are such that $P(A) = 0.43$, $P(B) = 0.48$, and $P(A \cup B) = 0.78$. Show that the events A and B are neither mutually exclusive nor independent. (3 marks)

5.1.12: (NECTA 2001) A bag contains 10 red balls, 9 blue balls, and 5 white balls. Three balls are taken from the bag at random and without replacement. Find the probability that all three balls are of the same colour. (3 marks)

5.2 Ordering and Choosing

Probability depends on the number of ways things are possible. The probability of something happening is the number of ways it can happen, divided by the total number of things that can happen. Thus, in this section, we will find how to count the number of ways things can happen.

5.2.1 Ordering

We begin with factorials, the number of ways of ordering n different things.

Definition: *The number of ways of arranging n different objects is given by $n!$, which is read* 'n factorial,' *and is given by the formula:*

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot (2) \cdot (1)$$

Why is this it? It all depends on the number of choices, the number of ways, the namba ya chaguzi. For example, if we have the letters A, B, and C, how many ways can we arrange these 3 objects?

Three is not too many, so we can just make a list:

$$\begin{array}{ccc} ABC & BAC & CAB \\ ACB & BCA & CBA \end{array}$$

There are 6 ways. But what if we look at the letters A, B, C, D, E, and F? There are far too many to just make a list, we need a way to calculate it.

However we do it, there will be a first letter, and second letter, and a third letter. How many choices are there for the first letter? If we have 6 letters, there are 6 choices for the first letter. Then, after picking the first letter, how many choices are remaining for the second letter? Zinabaki 5. We have already picked one for the first, so there are 5 choices remaining for the second letter. And now how many for the 3rd letter? 4 choices. And we can keep on going... The total number of possibilities is the product of all these choices. So, if we have 6 letters to arrange, we have $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$ ways to arrange them. I'm glad we didn't try to make a list! And before, when we had 3 items, our list showed 6 ways. Kumbe, $3! = 3 \cdot 2 \cdot 1 = 6$.

5.2.2 Permuting

5.2.3 Choosing

5.2.4 Doing All Three

5.2.5 Exercises

5.3 Basic Probability

5.4 Conditional Probability

5.4.1 Exercises

5.4.1: (NECTA 2005) A box contains 100 paper clips. 27 of the clips are too large and 16 of them are too small for the intended work. A paper clip is taken, judged, and not replaced. A second clip is then treated similarly. Calculate the probability that:
(a) Both paper clips are acceptable for the intended work.
(b) The first paper clip is too large and the second one is too small.
(c) One paper clip is too large and the other is too small. (10 marks)

5.5 Statistics

Statistics is a very important topic. In university, any science-related field, from medicine to forestry to engineering, requires a study of statistics. It is also used in business, for both marketing and quality control. The topics in the BAM syllabus cover just a brief introduction. It's a relatively easy topic, and well worth a little bit of time to learn it.

5.5.1 Data

If you ask 10 A-Level students, 'How old are you?' you may get responses like these: 18, 19, 20, 19, 23, 21, 18, 21, 21, 20. These are data.

Definition: Data*, which can be singular or plural, is a set of observations or measurements of a certain quality. A single number is a 'piece of data' or a 'datum'.*

By themselves, our data is not very useful, so we have several ways to make it more useful. A *frequency distribution table* shows, for each data point x, the number of times it occurs, called its *frequency*. For example, using the age data above, the frequency distribution table would be:

Age (x)	Frequency (f)
18	2
19	2
20	2
21	3
22	0
23	1

Definition: *The* sample size *is the number of data points, usually written n. The* relative frequency *for a certain data point is its frequency f divided by n. If you multiply relative frequency by 100%, the result is the percent frequency.*

If n is not given, an easy way to calculate it is to add all the frequencies. This is abbreviated $\Sigma f = n$. The capital Greek letter Sigma (Σ) means sum. For our data above, $n = 10$, which we can check by adding the frequencies: $\Sigma f = 2 + 2 + 2 + 3 + 0 + 1 = 10$. We can add these columns to our table:

Age (x)	Freq (f)	Relative Freq (f/n)	Percent Freq ($f/n \cdot 100\%$)
18	2	0.2	20%
19	2	0.2	20%
20	2	0.2	20%
21	3	0.3	30%
22	0	0.0	0%
23	1	0.1	10%

If there is lots of data, it can be too much to put into a frequency distribution table like this. In this case, we group the data into different *classes*. A class covers several data points, and we usually want an equal number of data points in each class. For the ages listed above, we could make classes such as $18 - 19$, $20 - 21$, and $22 - 23$, where two ages are in each class, or we could put 18-20 and 21-23, with 3 ages in each class. Here is a diagram for the first option:

Age (x)	Frequency (f)
$18 - 19$	4
$20 - 21$	5
$22 - 23$	1

The frequencies just add. This is how we organize our data. There are two main ways we analyse data: measures of central tendency and measures of dispersion.

5.5.2 Measures of Central Tendency

Measures of central tendency tell us about the center of data. The three common measures of central tendency are *mean*, *median*, and *mode*.

Definition: *The* mean*, or* average *of a data set is the sum of all observations divided by the total number of observations.*

The mean as calculated from data is usually written as $\bar{x}$. If the true mean is known it can be written as μ, but in most cases a given mean is based on incomplete data, so $\bar{x}$ is used. If there are n data points, called x, then

$$\bar{x} = \frac{\sum x}{n},$$

The mean is the sum of all data points divided by the number of data points. If you are using a frequency distribution table, then each x^i occurs f_i times, and $n = \sum f_i$, the total of the frequencies is the number of points, so

$$\bar{x} = \frac{\sum(x_i \cdot f_i)}{\sum f_i}$$

Pick an assumed mean A, in columns $x_i - A$ — $x_c = x_i - A/c$divide by class size — f — product $x_c \cdot f$. Then, $\bar{x}_c = \frac{\sum x_c \cdot f}{\sum f}$. Decoding, same steps reverse order. Multiply by class size, then add A.

Definition: *The* mode *of a data set is the value that occurs most often, the data point with the highest frequency.*

It is possible for a data set to have more than one mode.

Definition: *The* median *is the value in the middle when the data is arranged in ascending or descending order. If there is not a single middle term, then the two terms in the middle are averaged to calculate the median.*

Ex 1: *A certain data set is 7, 8, 12, 19, 6, 22, 9. What is the median?*
Solution: First we order the data (ascending or descending, both are fine).

$$6,\ 7,\ 8,\ 9,\ 12,\ 19,\ 22$$

Then we find the one in the middle: 9 is the median. ✠

However, if there is an even number of data points we need to average to find the median.

Ex 2: *A data set is 3, 7, 2, 3, 9, 6. Find the median.*
Solution: Writing it in order:

$$2,\ 3,\ 3,\ 6,\ 7,\ 9$$

When you find the middle, it's between 3 and 6, so the median is the average of 3 and 6.

$$\frac{3+6}{2} = 4.5$$

The median is 4.5. ✠

In many cases, the median is the best measure of central tendency for understanding the data. This is because the mean is heavily influenced by outliers. If a single point if very high or very low, it can have a big effect on the mean, but almost no effect on the median.

5.5.3 Measures of Dispersion

Measures of dispersion tell about the distribution of the data. Maybe it is all together, maybe it is spread out... The two measures of dispersion we will use are *range* and *standard deviation.*

Definition: *The* range *of a data set is the difference between the highest and lowest points.*

Ex 3: *The scores on a certain BAM exam are as follows: 37, 44, 46, 49, 54, 58, 63, 64, 64, 65, 71, 74, 83, 91. What is the range of the scores?*

Solution: The highest score is 91, the lowest score is 37, the highest score is 91, so the range is $91 - 37 = 54$. ✠

Definition: *The* standard deviation *of a data set is the average difference from the mean.*

To calculate the standard deviation, σ, first you must find the mean, $\bar{x}$. Then, for every point, you take it's distance from the mean, $x - \bar{x}$, and square it, $(x - \bar{x})^2$. Add up all of these, $\sum(x_i - \bar{x})^2$, divide by n, and take the square root. The easiest formula is:

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - (\bar{x})^2}$$

The sum $\sum x_i^2$ is the sum of the squares of all the data points. We divide this by n to get the average of the squares. Then we subtract the mean squared. Take the square root, and we get the standard deviation.

The purpose of the squaring is to make all the differences positive. But the end result is the average distance of a data point to the mean. Thus a large standard deviation tells you that the data is spread out, and a small standard deviation tells you that the data is all tight together.

Exercises

5.5.1: (NECTA 2008) The following table shows the marks of 100 students at Mzumbe University.

Marks	Frequency
60-62	5
63-65	18
66-68	42
69-71	27
72-74	8

(a) Find the
 i. Mean score.
 ii. Standard deviation.
(b) Draw the cumulative frequency and hence estimate the median. (10 marks)

5.5.2: (NECTA 2006) You are provided with the following frequency distribution table:

i	1	2	3	4	5
x_i	13	14	15	16	17
f_i	1	4	12	2	1

(a) Find the value of

i. $$\frac{\sum_{i=1}^{5}(x_i f_i)}{\sum_{i=1}^{5} f_i}$$

ii. $$\sqrt{\frac{\sum_{i=1}^{5}(x_i - \bar{x})^2 f_i}{\sum_{i=1}^{5} f_i}},$$

where $\bar{x}$ is your answer in (a) i.
(b) Find the
i. Median of the frequency distribution.
ii. Mode of the frequency distribution. (10 marks)

5.5.3: (NECTA 2005) The terminal marks in a Basic Applied Mathematics examination obtained by 40 students in one of the secondary schools in Tanzania are as follows:

66,	87,	79,	74,	84,	72,	81,	78,	68,	74,
80,	71,	91,	62,	77,	86,	87,	72,	80,	77,
76,	83,	75,	71,	83,	67,	94,	64,	82,	78,
77,	67,	76,	82,	78,	88,	66,	79,	74,	64.

From above data:
(a) Prepare a frequency distribution table with the lowest class interval of 60-64.
(b) Calculate the mean mark by using the coding method.
(c) Calculate the standard deviation correct to 2 decimal places. (10 marks)

5.5.4: (NECTA 2003) The marks obtained by 2 students in a certain examination were tabulated as follows:

Marks	50	65	70	75	80	85	90	95
Frequency	4	3	10	4	1	2	1	1

(a) Determine
i. The mean.
ii. The frequency.
(b) Determine the cumulative frequency, then draw an ogive.
(c) Compute the mean and standard deviation to two decimal places of the following distribution by using the coding method. Take $A = 170$.

Value	150	155	160	165	170	175	180	185	190
Frequency	5	11	13	17	26	21	9	5	3

(10 marks)

5.5.5: (NECTA 2002) The following table summarises the masses measured to the nearest kg of 200 animals of the same species:

Mass (kg)	Number of animals
75-79	7
80-84	30
85-89	66
90-94	57
95-99	27
100-104	13

Calculate:
(a) The mean using a deviation approach. **(b)** The standard deviation of masses correct to two decimal places. (10 marks)

5.5.6: (NECTA 2001) Consider the following distribution table:

Class Mark	7	12	17	22	27	32
Frequency	4	9	16	22	6	3

Determine
(a) The mean. **(b)** The mode. **(c)** The median.
(6 marks)

5.5.7: (NECTA 2000) The table below shows the IQs of 480 children at a certain school.

IQ	70	74	78	82	86	90	94	98	102	106	110	114	118	122	126
No. of Children	4	9	16	28	45	66	y	72	54	38	27	18	11	5	2

(a) Find the value of y.
(b) Calculate the mean and standard deviation using an assumed mean $A = 94$. (6 marks)

Chapter 6

Miscellaneous Other Topics

The topics in this chapter are mostly unrelated to each other. They can be studied individually at most any time in the course.

6.1 $\sqrt{\text{Roots}}$ of Quadratics

A polynomial $P(x)$ has *roots*. The roots are the values of x for which $P(x) = 0$. In the case of quadratics, you know very well how to find the roots by either factoring or using the quadratic formula.

Definition: *The* roots *of a polynomial $P(x)$ are the values of x for which $P(x) = 0$. They are the solutions of the equation $P(x) = 0$.*

A polynomial of degree n has at most n different, real roots. Remember that the *degree* of a polynomial is the biggest exponent of x. We will look closely at quadratics. The focus of this section is learning about the roots without actually solving for them.

6.1.1 Real Roots and the Quadratic Formula

When we are dealing with real numbers ($\mathbb{R}$), we cannot take the square root of a negative number. (If you continue in math beyond BAM, you will learn about complex numbers, $\mathbb{C}$, where you can take square roots of negatives.) A quadratic $ax^2 + bx + c = 0$ has roots given by the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

thus its roots are real if and only if

$$b^2 - 4ac \geq 0.$$

$$\text{If} \quad b^2 - 4ac < 0, \quad \text{then there are no real roots.}$$

There are roots, but they are complex, not real.

Ex 1: *Show that $x^2 + 1 = 0$ has no real roots.*

Solution: We use the quadratic formula with $a = 1$, $b = 0$, and $c = 1$.

$$b^2 - 4ac = 0 - 4 \cdot 1 \cdot 1 = -4 < 0$$

Thus $x^2 + 1$ has no real roots. ✠

6.1.2 Manipulating Roots

Regardless of whether the roots of a quadratic are real of complex, we can still learn some things about them, and even construct equations with similar roots.

A polynomial with roots α and β may be written and expanded as

$$\begin{aligned}(x-\alpha)(x-\beta)&=0\\ x^2-\alpha x-\beta x+\alpha\beta&=0\\ x^2-(\alpha+\beta)x+\alpha\beta&=0\end{aligned}$$

So, if we take a general quadratic, ax^2+bx+c, and call the roots α and β, then we can say that

$$ax^2+bx+c=0 \quad \text{which can be written as}$$

$$\left.\begin{aligned}x^2+\frac{b}{a}x+\frac{c}{a}&=0\\ x^2-(\alpha+\beta)x+\alpha\beta&=0\end{aligned}\right\rangle \text{ These are the same!}$$

So, matching up the corresponding parts,

$$\alpha+\beta=-\frac{b}{a} \quad \text{and} \quad \alpha\beta=\frac{c}{a}$$

This is big information, and this one line is all you need to remember to be able to solve most problems.

Ex 2: *The roots of* $3x^2+4x-5=0$ *are* α *and* β*. Find the values of* ***(a)*** $1/\alpha+1/\beta$*, and* ***(b)*** $\alpha^2+\beta^2$*.*

Solution: It is a puzzle. We don't know α or β, but we do know $\alpha+\beta$ and $\alpha\beta$. We want to write $1/\alpha+1/\beta$ and $\alpha^2+\beta^2$ in terms of what we know. Starting with part (a):

$$\begin{aligned}1/\alpha+1/\beta&=\frac{\beta}{\alpha\beta}+\frac{\alpha}{\alpha\beta} && \text{Finding a common denominator}\\ &=\frac{\alpha+\beta}{\alpha\beta} && \text{Adding the fractions}\\ &=\frac{-b/a}{c/a} && \text{Substituting in what we know}\\ &=\frac{-b}{c} && \text{Simplifying}\end{aligned}$$

This is all done in general. Now that we have a general answer, $1/\alpha+1/\beta=\frac{-b}{c}$, we can substitute in the values of b and c from the quadratic given in the problem, $3x^2+4x-5=0$. We see that $b=4$ and $c=-5$, so $1/\alpha+1/\beta=\frac{-b}{c}=\frac{-4}{-5}=\frac{4}{5}$.

Now on to part (b). Our method is the same. We know $\alpha+\beta$ and $\alpha\beta$, we want to write $\alpha^2+\beta^2$ in terms of what we know.

$$\begin{aligned}\alpha^2+\beta^2&=\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta && \text{This is a neat trick}\\ &=\alpha^2+2\alpha\beta+\beta^2-2\alpha\beta && \text{Because } (a+b)^2=a^2+2ab+b^2\\ &=(\alpha+\beta)^2-2\alpha\beta && \text{Isn't that cool?}\\ &=(-b/a)^2-2(c/a) && \text{Substituting in what we know}\\ &=\frac{b^2}{a^2}-2\frac{c}{a} && \text{Simplifying}\end{aligned}$$

And, like before, now that we are done with variables we can substitute in our values $a = 3$, $b = 4$, and $c = -5$. Thus

$$\alpha^2 + \beta^2 = \frac{b^2}{a^2} - 2\frac{c}{a} = \frac{4^2}{3^2} - 2\frac{-5}{3} = \frac{16}{9} + \frac{10}{3} = \frac{46}{9}.$$

✠

This is the most common style of problem you will see on this topic. The procedure is always the same. First, using the variables α and β, find a way to write what you want to know in terms of what you know. Then substitute in to get the answer.

Ex 3: *The roots of* $2x^2 - 7x + 4 = 0$ *are* α *and* β*. Find an equation with integer coefficients whose roots are* α/β *and* β/α*.*

Solution: We *don't* need to find α/β and β/α, rather what we need is the equation $x^2 - (\frac{\alpha}{\beta} + \frac{\beta}{\alpha})x + \frac{\alpha}{\beta}\frac{\beta}{\alpha} = x^2 - (\frac{\alpha}{\beta} + \frac{\beta}{\alpha})x + 1$. So, really all we need is $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

$$\begin{aligned}
\alpha/\beta + \beta/\alpha &= \frac{\alpha^2}{\alpha\beta} + \frac{\beta^2}{\alpha\beta} && \text{Finding a common denominator} \\
&= \frac{\alpha^2 + \beta^2}{\alpha\beta} && \text{Adding the fractions} \\
&= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{\alpha\beta} && \text{Using the same trick as in part (b) above.} \\
&= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} && \text{It's a very useful trick.}
\end{aligned}$$

We know that $\alpha + \beta = -\frac{-7}{2} = \frac{7}{2}$ and that $\alpha\beta = \frac{4}{2} = 2$. Thus $\alpha/\beta + \beta/\alpha = \frac{(\frac{7}{2})^2 - 2\cdot 2}{2} = \frac{33}{8}$. And now our equation is $x^2 + \frac{33}{8}x + 1 = 0$. Unfortunately this doesn't have integer coefficients! But, if we multiply through by 8, then we get $8x^2 + 33x + 8 = 0$, which does.

Note: We can multiply by any constant without changing the roots. We used 8 here because it is the easiest, but other answers like $16x^2 + 66x + 16 = 0$ and $24x^2 + 99x + 24 = 0$ are just fine too. ✠

Exercises

6.1.1: (NECTA 2002) Show that $2x^2 - 3x + 4 = 0$ has no real roots. (2 marks)

6.1.2: (NECTA 2002) If $3x^2 - 6x + 8 = 0$ has roots α and β, find the equation whose roots are $1/\alpha$ and $1/\beta$. (4 marks)

6.1.3: (NECTA 2003) Given that α and β are roots of the function $f(x) = ax^2 + bx + c$, where a, b, and c are real and $a \neq 0$, **(a)** write down the values of $\alpha + \beta$ and $\alpha\beta$ in terms of a, b, and c. **(b)** State the conditions that the roots α and β in (a) above are equal in magnitude but opposite in sign. (2 marks)

6.2 Linear Programming

Linear Programming is a technique used to decide what is the best way to act in certain types of situations. Linear programming can be used when we are choosing values of several variables, subject to linear constraints, and our goal is to maximize or minimize a linear function of those variables, called a metric. Lets consider an example so that we can see what this means.

Ex 1: *A school is trying to plan meals for its students. A student requires 20g of protein and 200g of carbohydrates each day to be healthy. The school will feed the students beans and rice. Beans contain 360g of protein and 400g of carbohydrates per kilogram. Rice contains 10g of protein and 900g of carbohydrates per kilogram. Beans cost 1400/= per kilogram, and rice costs 1000/= per kilogram. What quantity of beans and of rice should the school give to each student in order to feed them adequate food for the lowest cost?*

Solution:

In this problem, we are choosing values of two variables. Our two variables are the quantity of beans and the quantity of rice to give each student. These variables are subject to linear constraints, because the total amount of protein (which is a linear function of the two variables, quantity of beans and quantity of rice) must be above a certain level, and the total amount of carbohydrates (also a linear function of our two variables) must be above a certain level as well. Finally, our goal is to minimize the total cost, which is a linear function of our variables. So we see that this is the sort of problem that linear programming can be used to solve.

Linear programming questions are usually presented like this, as a paragraph of writing. Once we've identified it as a problem that we can use linear programming to solve, our first task is to identify our variables, our constraints, and our metric.

Identify the variables

Let us call our two variables B, for the quantity of beans given to each student, and R, for the quantity of rice given to each student. Our variables have now been identified.

Identify the constraints

We have been told that each student needs 20g of protein a day. How much protein does each student receive? A kilogram of beans contains 150g of protein. So if a student receives B kilograms of beans, she receives $B \cdot 360\text{g}$ of protein from beans. Likewise, if she receives R kilograms of rice, she receives $R \cdot 10\text{g}$ of protein from rice. Her total protein is then $B \cdot 360\text{g} + R \cdot 10\text{g}$. This must be at least 20g, so we have one of our constraints $B \cdot 360\text{g} + R \cdot 10\text{g} \geq 20\text{g}$. Similarly, the students receives 400g of carbohydrates for each kilogram of beans and 900g of carbohydrates for each kilogram of rice. Thus we have a second constraint that $B \cdot 400\text{g} + R \cdot 900\text{g} \geq 200\text{g}$. We should also add that $B \geq 0$ and $R \geq 0$, as it is not possible for a student to eat a negative quantity of food. We have now identified our four constrains. Note that these four constraints are all *linear constraints*. That is to say that each variable can by multiplied by a constant factor, but never squared, logged, multiplied by another variable, or anything like that. If our constraints are not linear constraints, we will not be able to use linear programming to solve the problem.

Identify the metric

Finally, let's consider our metric. We want to minimize the amount of money spent. The amount of money spent on beans will be $B \cdot 1400/=$ and on rice $R \cdot 1000/=$. Thus, the total amount of money spent per student will be $B \cdot 1400/= + R \cdot 1000/=$. We want to minimize this number. Thus, we have identified our metric. Note that this is a linear metric. If our metric is not linear, we can't use linear programming to solve the problem.

Draw a sketch

Let's draw a quick sketch to better understand our constraints. We draw a pair of axes using our two variables, B and R. Our first constraint is $B \cdot 150\text{g} + R \cdot 50\text{g} \geq 20\text{g}$. We draw the line $B \cdot 360\text{g} + R \cdot 10\text{g} = 20\text{g}$, and shade in the region below. This shaded region does not satisfy our constraint, and thus none of the points in this region is a possible solution. Next we draw the line $B \cdot 400\text{g} + R \cdot 900\text{g} = 200\text{g}$ and shade the region below that for our second constraint.

We also draw the line $B = 0$ and shade the region to its left, and the line $R = 0$ and shade the region below it. We can now see the remaining region of allowable solutions. In this case, it's in the upper right-hand part of our sketch.

Find the corners

The secret to linear programming problems is that *the best solution is always a corner of the boundary* of allowable solutions. In this case, there are three corners. One is where the B axis intersects the first constraint (point A in our sketch). One is where the second constraint intersects the R axis (point B in our sketch). One is where the first and second constraints intersect each other (point C in our sketch). Note that there points D and E are outside of the region of allowable solutions, so we can ignore them. Solving for the locations of these points, we find:

Point A

$$\begin{aligned}
B \cdot 360\text{g} + R \cdot 10\text{g} &= 20\text{g} && \text{First constraint} \\
B &= 0 && \text{Third constraint} \\
0 \cdot 360\text{g} + R \cdot 10\text{g} &= 20\text{g} && \text{Substituting the 3rd constraint into the 1st} \\
R &= \frac{20\text{g}}{10\text{g}} && \text{Solving} \\
(B, R) &= (0, 2) && \text{Point A}
\end{aligned}$$

Point B

$$\begin{aligned}
B \cdot 400\text{g} + R \cdot 900\text{g} &= 200\text{g} && \text{Second constraint} \\
R &= 0 && \text{Fourth constraint} \\
B \cdot 400\text{g} + 0 \cdot 900\text{g} &= 200\text{g} && \text{Substituting the 4th constraint into the 2nd} \\
B &= \frac{200\text{g}}{400\text{g}} && \text{Solving} \\
(B, R) &= (0.5, 0) && \text{Point B}
\end{aligned}$$

Point C

$$\begin{aligned}
B \cdot 400\text{g} + R \cdot 900\text{g} &= 200\text{g} && \text{Second constraint} \\
B \cdot 3600 + R \cdot 8100\text{g} &= 1800\text{g} && \text{Second constrain times 9} \\
B \cdot 360\text{g} + R \cdot 10\text{g} &= 20\text{g} && \text{First constraint} \\
B \cdot 3600\text{g} + R \cdot 100\text{g} &= 200\text{g} && \text{1st constraint times 10} \\
B \cdot 0\text{g} + R \cdot 8000\text{g} &= 1600\text{g} && \text{9 times 2nd minus 10 times 1st} \\
R \cdot 8000\text{g} &= 1600\text{g} \\
R &= 0.2 \\
B \cdot 360\text{g} + 0.2 \cdot 10\text{g} &= 20\text{g} && \text{Substituting R into 1st constraint} \\
B \cdot 360\text{g} &= 18\text{g} \\
B &= \frac{18\text{g}}{360\text{g}} \\
B &= 0.05 \\
(B, R) &= (0.05,\ 0.2) && \text{Point C}
\end{aligned}$$

Evaluate the Metric at Each Corner
Because in linear programming problems *the best solution is always a corner of the boundary*, we know that the best solution is either point A, B, or C. So we calculate our metric at each of these points. At point A, $B = 0$ and $R = 2$; A represents 2 kilograms of rice and no beans. The cost of this is $B \cdot 1400/ = +R \cdot 1000/ =$, or 2000/=. At point B, B is 0.5 and R is 0; B represents 0.5 kilograms of beans and no rice. The cost of this is 700/=. At point C, B is 0.05 and R is 0.2; C represents 50g of beans and 200g of rice. The cost of this is 270/=. We see that point A has a cost of 2000/=, point B of 700/=, and point C of 270/=. As our goal is to minimize the cost, we choose point C. ✠

Exercises

6.2.1: A smuggler is transporting two types of supplies to a black-market purchaser: plutonium and human kidneys. He is riding his bicycle. If the mass of plutonium exceeds 10kg the plutonium will melt down, which is unsafe. The kidneys must be stored in a special cooler and rushed to the purchaser at high speed or they will spoil. The cooler can hold 10kg of kidneys. If the total mass of the supplies exceeds 15kg the smuggler will not be able to ride his bicycle fast enough, and the kidneys will spoil. If the smuggler can earn a profit of 10,000 euros per kg of plutonium, and 15,000 euros per kg of kidneys, what is his maximum profit?

6.2.2: A certain daladala can take two kinds of passengers. Mamas weigh 100kg, carry 10kg luggage, and pay 500/=. Students weigh 50kg, carry 20kg luggage, and pay only 400/=. All passengers sit in the vehicle and all luggage is strapped to the roof. If the total weight in the vehicle exceeds 2500kg, the floor of the daladala will collapse on the highway, killing everyone. If the total weight on the roof exceeds 500kg, the roof will collapse, killing everyone. If the total number of passengers exceeds 30, they will riot and kill the driver and the conductor. How many of each type of passenger should the conductor choose to maximize his profit without dying?

6.2.3: A safari company offers two kinds of safaris, driving and walking safaris. A driving safari requires a car and one employee to drive. A walking safari requires two employees, one guide and one porter to carry lunch boxes. The guide must be armed with a rifle so that the tourists won't be too afraid of animals. The company earns a profit of 30,000/= for each driving safari and 80,000/= for each walking safari. If the company has 6 cars, 6 rifles, and 16 employees, what is the maximum profit it can earn?

6.2.4: A student calculates that each cup of tea that she drinks immediately before her exam will improve his score by 3 marks, and each cup of coffee she drinks before her exam will improve her score by 5 marks. Each cup of tea has 60g sugar and 40mg caffeine. Each cup of coffee has 20g sugar and 80mg caffeine. If the student consumes more than 240g of sugar, she will go into diabetic shock and fail her exam. If the student consumes more than 400mg of caffeine, she will suffer from heart palpitations and fail her exam. If the student consumes more than 6 cups of fluid total her bladder will rupture and she will fail her exam. How many cups of each coffee and tea should she drink to maximize her score?

6.2.5: An mchawi is payed by clients to put curses on victims. A Jealous Wife will pay 40,000/= to have her Cheating Husband made impotent. This requires 15g of Albino Eye, 20g of Little Child Brain, and 1 cauldron. A Frustrated Teacher will pay 50,000/= to have a NECTA Employee turned into a slug. This requires 5g of Albino Eye, 40g of Little Child Brain, and 1 cauldron. The mchawi has only 120 g of Albino Eye , 320g of Little Child Brain, and 10 cauldrons. What is the maximum amount of money he can earn?

6.2.6: (NECTA 2008) A firm makes two types of furniture: chairs and tables. The contribution of each product as calculated by the accounting department is 20/= and 30/= per chair and per table, respectively. Both products are processed on three machines M_1, M_2, and M_3. The time required in hours per week on each machine is as follows (see table). How should the

Machine	Chair	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

manufacturer schedule his production in order to maximize contribution? (10 marks)

6.2.7: (NECTA 2006) A certain farmer wants to use part of his shamba to plant cabbages and potatoes. He divides that part of his shamba into several equal-sized portions. The farming of cabbages will cost Sh. 24,000 per portion, and potatoes 8,000 per portion. The maximum (**note:** the original question says 'minimum' here, but that makes no sense. Corrected to 'maximum' by author.) funds which can be used in farming the two crops are Sh. 240,000. Cabbages require 10 man hours per portion, while potatoes require 200 man hours per portion. The estimated profit is Sh. 16,000 per portion of cabbages and Sh. 12,000 per portion of potatoes.
(a) How should he allocate the expected shamba for maximum profit?
(b) What is the maximum profit? (10 marks)

6.2.8: (NECTA 2005) A person requires 10, 12, and 12 units of mineral elements A, B, and C, respectively, for her diet. A liquid diet contains 5, 2, and 1 units of A, B, and C, respectively, per can; and a dry diet contains 1, 2, and 4 units of A, B, and C, respectively, per carton. If the liquid diet is sold at the price of Sh. 3,000 per can and the dry diet is sold at the price of Sh. 2,000 per carton, how many cans and cartons should a person purchase to minimize the costs and meet the dietary requirements? (10 marks)

6.2.9: (NECTA 2003) By shading the unrequired part, show the region represented by the following set of inequalities:

$$y \leq 2x, \qquad x + y \leq 6, \qquad x + y \geq 3, \qquad y \geq 0, \qquad x \geq 0, \qquad x \leq 5.$$

(4 marks)

6.2.10: (NECTA 2003) (a) Shade the unrequired region of the following inequalities:

$$\begin{aligned} 2x + y &\leq 10 \\ 4x + 3y &\leq 24 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

(b) Find the maximum value of $f(x, y) = 2x + 4y$ in the reqion required in (a) above. (7 marks)

6.2.11: (NECTA 2002) Students are about to take a test that contains questions of type A worth 10 points and questions of type B worth 25 points. They must do at least 3 questions of type A but not more than 12. They must do at least 4 questions of type B but not more than 15. In total they cannot do more than 20 questions. How many of each type of question must a student do to maximize the score? What is the maximum score? (10 marks)

6.2.12: (NECTA 2001) Graph the feasible set for the system of inequalities:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ 4x - 4y &\geq -4 \end{aligned}$$

(3 marks)

6.3 Sequences$_1$ and$_2$ Series$_3$

Definition: *A* sequence *is a list—a set with a certain order—that follows a rule or pattern.*

Definition: *A* series *is a sequence where the terms are added together.*

For example, a good sequence is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

and its corresponding series is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

This particular series is called the *Harmonic Series*.

Ex 1: *Find the rules for the following sequences:*
(a) $1, 2, 3, 4, \ldots$
(b) $2, 4, 6, 8, \ldots$
(c) $1, 3, 9, 27, \ldots$
Solution: **(a)** The rule is to add 1 each time.
(b) The rule is to add 2 each time.
(c) The rule is to multiply by 3 each time. ✠

Can you find the rules to these sequences?
$1, 1, 2, 3, 5, 8, 13, \ldots$
M, M, T, N, T, S, S,...
Take a minute and really try before reading on to the answer. They will seem very clear when you learn the rule. Can you find the next term yourself?

The first sequence is called the *Fibonacci Sequence*. The first two terms are 1, and then each term is the sum of the previous two. $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, $5 + 8 = 13$, so the next term is 21 because $8 + 13 = 21$. The second sequence is a puzzle. Think about it, maybe you will find the answer. I will tell you that, if you find the answer, you will have no doubt, *bila shaka*, that your answer is correct.

Sometimes the rule is obvious, sometimes it isn't. Mostly we will deal with sequences that have easy rules. They come in two types: Arithmetic Progressions and Geometric Progressions.

• Note $\mathcal{O}$n $\mathcal{N}$otation •

We use the Greek capital letter sigma, Σ, to indicate a sum. Usually it will look something like this:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i,$$

the right-hand side is read 'the sum from i=1 to $i = n$ of A_i.

Ex 2: *Write* ***(a)*** $a_1 + a_2 + a_3 + a_4 + a_5$, *and* ***(b)*** $a_{23} + a_{24} + a_{25} + \cdots + a_{100}$ *in sigma sum notation.*
Solution: **(a)**

$$a_1 + a_2 + a_3 + a_4 + a_5 \sum_{i=1}^{5} a_i$$

(b)

$$a_{23} + a_{24} + a_{25} + \cdots + a_{100} \sum_{i=23}^{100} a_i$$

✠

Definition: *An* arithmetic progression (A.P.) *is a sequence where there is a common difference between successive terms.*

If the common difference is d, then we can write

$$a_{n+1} = a_n + d,$$

which means that if you have a term a_n, to find the next term, you add d. If the first term is a_1, then the sequence is

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \dots$$

and the n^{th} term is given by

$$a_n = a_1 + (n-1)d.$$

The formula is (n-1) because the $\mathbf{1}^{st}$ term is $a_1 + \mathbf{0}d$, the $\mathbf{2}^{nd}$ term is $a_1 + \mathbf{1}d$, the $\mathbf{3}^{rd}$ term is $a_1 + \mathbf{2}d$, etc.

The sum of the first n terms of an A.P. is given by

$$S_n = \sum_{i=1}^{n} a_i = \frac{n}{2}\big(a_1 + a_n\big).$$

If we substitute in $a_n = a_1 + (n-1)d$ from above, this becomes

$$S_n = \sum_{i=1}^{n} a_i = \frac{n}{2}\big[2a_1 + (n-1)d\big].$$

Notice for the series $1 + 2 + 3 + \cdots + n$ that $a_1 = 1$, $d = 1$, so $S_n = n(n+1)/2$.

Ex 3: *Identify the first term, a_1, the common difference d, and find the sum of the first 10 terms for the following sequences:*

(a) $1, 2, 3, 4, \dots$

(b) $7, 4, 1, -2, \dots$

Solution: **(a)** The first term $a_1 = 1$, the common difference is 1, so the sum of the first 10 terms is given by

$$S_{10} = \frac{n}{2}\big[2a_1 + (n-1)d\big] = \frac{10}{2}\big[2 \cdot 1 + (10-1) \cdot 1\big] = 55.$$

(b) The first term $a_1 = 7$, the common difference is -3, so the sum of the first 10 terms is

$$S_{10} = \frac{n}{2}\big[2a_1 + (n-1)d\big] = \frac{10}{2}\big[2 \cdot 7 + (10-1) \cdot (-3)\big] = -65.$$

✠

Ex 4: *An arithmetic progression has first term 3 and 14th term 55. Find the common difference and the sum of the first 14 terms.*

Solution: First we'll find the common difference, d. What we know is that $a_1 = 3$ and $a_{14} = 55$. But we also know that $a_{14} = a_1 + (14-1)d$. Thus

$$a_{14} = a_1 + (14-1)d \tag{6.1}$$
$$55 = 3 + 13d \tag{6.2}$$
$$52 = 13d \tag{6.3}$$
$$d = 4 \tag{6.4}$$

As for the sum, we don't even need to know d.

$$\begin{aligned} S_{14} &= \frac{14}{2}(a_1 + a_14) \\ &= 7(3+55) \\ &= 406 \end{aligned}$$

✠

Definition: *A* geometric progression (G.P.) *is a sequence where there is a common ration between successive terms.*

If the common ratio is r, then

$$a_{n+1} = ra_n,$$

to find the next term, just multiply by r. If the first term is a_1, then the sequence is

$$a_1, ra_1, r^2a_1, r^3a_1, \ldots$$

and the n^{th} term is given by

$$a_n = r^{n-1}a_1.$$

The sum of the first n terms of a G.P. is given by

$$S_n = \sum_{i=1}^{n} a_i = a_1 \frac{1-r^n}{1-r}$$

for any $r \neq 1$. (If $r = 1$ then the sequence is $a_1, a_1, a_1, \ldots$, and the series $a_1 + a_1 + \cdots + a_n = na_1$.)

Ex 5: *Identify the first term, a_1, the common ratio r, and find the sum of the first 10 terms for the following sequences:*
(a) $2, 6, 18, 54, \ldots$
(b) $1/2, 1/4, 1/8, 1/16, \ldots$

Solution: **(a)** The first term is $a_1 = 2$, the common ratio $r = 3$, so the sum for the first 10 terms:

$$S_n = a_1 \frac{1-r^n}{1-r} = 2\,\frac{1-3^{10}}{1-3} = 2\,\frac{1-59049}{-2} = 59048$$

(b) The first term $a_1 = 1/2$, the common ration is $r = 1/2$, so the sum is:

$$S_{10} = \frac{1}{2}\cdot\frac{1-(1/2)^{10}}{1-(1/2)} = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{1024} = 0.99902348$$

✠

Definition: *Some geometric series are* convergent, *if we take the limit as $n \to \infty$, we will get a real number. Others are* divergent, *they do not approach any specific value.*

A geometric series with common ratio r is convergent if $|r| < 1$. If $|r| \geq 1$ then the series is divergent. If geometric series is convergent, then its sum is

$$\sum_{i=1}^{\infty} a_i = \frac{a}{1-r}$$

Exercises

6.3.1: (NECTA 2005) The sum of the first and fifth terms of an arithmetic progression is 18, while the fifth term is 6 more than the third term. Find the sum of the first 10 terms. (3 marks)

6.3.2: (NECTA 2005) A biscuit factory starts producing biscuits at the rate of 50,000 per hour. This rate decreases by 10% every hour. Calculate the total number of biscuits produced in the first 3 hours. (3 marks)

6.3.3: (NECTA 2001) The 9^{th} term of an A.P. is twice as great as the third term, and the 15^{th} term is 27. Find the sum of the first 25 terms of the series. (4 marks)

6.3.4: (NECTA 2001) Find the seventh term of an A.P. whose first term, second term, and fifth term are in a G.P. and whose first term is 2. (2 marks)

6.3.5: (NECTA 2001) Show that $\log_3 x + \log_9 x + \log_{81} x + \ldots$ is a geometric progression with a common ration of $1/2$. (3 marks)

6.3.6: (NECTA 2000) In a geometrical progression the common ration is 2. Find the value of n for which the sum of $2n$ terms is 33 times the sum of n terms. (2 marks)

6.3.7: (NECTA 2000) The first term of an A.P. is -12 and the last term is 40. If the sum of the progression is 196, find the number of terms and the common ratio. (2 marks)

6.4 Pascal's Triangle

Pascal's Triangle is like a magic trick. It is also very 'deep,' it shows how different topics are connected. It really is just a kind of series, but it connects probability with algebra, and has other applications as well.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\text{etc.}
\end{array}$$

The start and end of each row is a 1. For the middle of a row, each number is the sum of the two numbers above it. For example, the row after 1 3 3 1 begins with 1, then the next number is $1 + 3 = 4$, then $3 + 3 = 6$, then $3 + 1 = 4$, and it ends with 1.

Now, let's look at $(a + b)^n$.

$$\begin{array}{rcr}
(a+b)^0 = & 1 & n = 0 \\
(a+b)^1 = & 1a{+}1b & n = 1 \\
(a+b)^2 = & 1a^2 + 2ab + 1b^2 & n = 2 \\
(a+b)^3 = & 1a^3 + 3a^2b{+}3ab^2 + 1b^3 & n = 3 \\
(a+b)^4 = & 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 & n = 4 \\
(a+b)^5 = & 1a^5 + 5a^4b + 10a^3b^2{+}10a^2b^3 + 5ab^4 + 1b^5 & n = 5 \\
(a+b)^6 = & 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6 & n = 6
\end{array}$$

And last, we'll look at ${}^nC_r$, the choosing function from probability.

$$\begin{array}{rcr}
{}^0C_0 = & 1 & n = 0 \\
{}^1C_0,\ {}^1C_1 = & 1,\ 1 & n = 1 \\
{}^2C_0,\ {}^2C_1,\ {}^2C_2 = & 1,\ 2,\ 1 & n = 2 \\
{}^3C_0,\ {}^3C_1,\ {}^3C_2,\ {}^3C_3 = & 1,\ 3,\ 3,\ 1 & n = 3 \\
{}^4C_0,\ {}^4C_1,\ {}^4C_2,\ {}^4C_3,\ {}^4C_4 = & 1,\ 4,\ 6,\ 4,\ 1 & n = 4 \\
{}^5C_0,\ {}^5C_1,\ {}^5C_2,\ {}^5C_3,\ {}^5C_4,\ {}^5C_5 = & 1,\ 5,\ 10,\ 10,\ 5,\ 1, & n = 5 \\
{}^6C_0,\ {}^6C_1,\ {}^6C_2,\ {}^6C_3,\ {}^6C_4,\ {}^6C_5,\ {}^6C_6 = & 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1 & n = 6
\end{array}$$

Look at these until you see the pattern. The pattern is stated kwa kihisabati as the Binomial Expansion Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \left({}^nC_i a^{n-i} b^i \right)$$

Basically, it says that you take the numbers from Pascal's Triangle as the coefficients and that the powers of a are n, $n-1$, $n-2, \ldots$ 0, and the powers of b are 0, 1, $2, \ldots$ n.

Ex 1: *Write the binomial expansion of* $(x+3)^5$

Solution: From Pascal's Triangle, we see that the coefficients are 1, 5, 10, 10, 5, 1. (Note: you can always know you are on the correct row because the second coefficient is the exponent n. Here, we have $(x+3)^5$, and the second coefficient is 5, so it's good!) We'll use descending powers of x, so the powers of x will decrease and the powers of 3 will increase:

$$\begin{aligned}(x+3)^5 &= \\ &= 1x^5 \cdot 3^0 + 5x^4 \cdot 3^1 + 10x^3 \cdot 3^2 + 10x^2 \cdot 3^3 + 5x^1 \cdot 3^4 \\ &\quad + 1x^0 \cdot 3^5 x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243\end{aligned}$$

✠

Ex 2: *Write out the binomial expansion of* $(1-x)^7$ *in ascending powers of* x *until the term with* x^3*. Then approximate* 0.98^7*.*

Solution: Once again, we look to Pascal's Triangle for the coefficients. The row that has 7 in it goes 1, $7,:$ 21, $35, \ldots$, and that will be enough because we only need to go until x^3. Because of the minus x, we will treat it as $(1+(-x))^7$.

$$\begin{aligned}(1-x)^7 &= \\ &= 1 \cdot 1^7 \cdot (-x)^0 + 7 \cdot 1^6 \cdot (-x)^1 + 21 \cdot 1^5 \cdot (-x)^2 + 35 \cdot 1^4 \cdot (-x)^3 + \cdots \\ &= 1 - 7x + 21x^2 - 35x^3 + \cdots\end{aligned}$$

Then, to approximate $(0.98)^7$, we see that $0.98 = 1 - 0.02$. So we put in 0.02 for x, and we get that

$$\begin{aligned}0.98^5 &= (1-0.02)^7 \\ &\approx 1 - 7(0.02) + 21(0.02)^2 - 35(0.02)^3 + \cdots \\ &\approx\end{aligned}$$

✠

Exercises

6.4.1: Write out Pascal's Triangle until the line that begins '1 8 28 ...'.

6.4.2: Write the binomial expansions for the following:
(a) $(a+b)^6$
(b) $(1+x)^6$
(c) $(1-x)^6$

6.4.3: Write the first 4 terms of the binomial expansion of $(1+x)^7$ in ascending powers of x. Hence approximate $(1.1)^7$ correct to 3 significant figures.

6.4.4: (NECTA 2003) Expand $(x-1)^5$ in ascending powers of x up to the term x^3, hence use the expansion to evaluate $(0.95)^5$ to three significant figures. *Note:* This does not work. Do not use $(x-1)^5$; instead use $(1-x)^5$. (2 marks)

6.4.5: (NECTA 2001) Write the first 3 terms of the binomial expansion of $(1+x)^6$. Hence approximate $(1.001)^6$ correct to 4 significant figures. (2 marks)

6.5 Set Theory

Definition: *A* set *is a group of items, called* elements. *A set has no order.*

The elements of a set are written inside curly braces: { }. Sets are usually called with capital letters, like A or B. Sets can have numbers or anything else for elements.

Definition: *The* cardinality *of a set is the number of elements. The cardinality of a set A is written as $|A|$ or as $n(A)$.*

Examples of sets include: $A = \{1, 2, 3, 4\}$, $B = \{x : x > 0\}$, and $C = \{1, \text{Elephant}, \heartsuit\}$. In these cases, $|A| = 4$, $|B|$ is infinite, and $|C| = 2$.

The symbol $\in$ means 'in,' and $\notin$ means 'not in.' For example, using A above, we could say $1 \in A$, 1 is in A, but $6 \notin A$, 6 is not in A.

Definition: *The* union *of 2 sets is a set that includes all elements that are in* either *of the 2 sets.*

Union is written as $\cup$. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

Definition: *The* intersect *of 2 sets is a set that includes all elements that are in* both *of the 2 sets.*

Intersect is written as $\cap$. Using the same $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$,

$$A \cap B = \{3, 4\}.$$

A useful formula for cardinality is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex 1: *If 12 students say they like Fiddy Q, and 7 students say they like Banana, and there are 15 students total, how many like both Fiddy Q and Banana?*

Solution: We let A be the set of students who like Fiddy Q, and B be the set of students who like Banana. The data tells us that the total number of students, $|A \cup B| = 15$, and what we want to find is the number of students who like *both* Fiddy Q and Banana, $|A \cap B|$.

$$\begin{aligned} |A \cup B| &= |A| = |B| - |A \cap B| \\ 15 &= 12 + 7 - |A \cap B| \\ 15 &= 19 - |A \cap B| |A \cap B| && = 4 \end{aligned}$$

4 students like both Fiddy Q and Banana. ✠

Definition: *The* null set *or* empty set *is the set with no elements. It is written as $\varnothing$.*

If $C = \{1, 2, 3\}$ and $D = \{4, 5, 6\}$, then $C \cap D = \varnothing$.

Definition: *The* complement *of a set is everything* not *in the set.*

The complement of A is written as A^C or A'.

De Morgan's Laws:

$$(A \cap B)' = A' \cup B' \qquad \text{and} \qquad (A \cup B)' = A' \cap B'$$

Distributive Laws also hold:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

Exercises

6.5.1: (NECTA 2002) In a certain school, there is an equal number of boy and girl students. 1/4 of the boys and 1/10 of the girls go to school on foot. 1/3 of the boys and 1/2 of the girls go to school by bicycle, and the rest go by daladala. Find the proportion of students:
(a) That are girls who go by daladala.
(b) That go by daladala. (10 marks)

6.5.2: (NECTA 2001) 64 Students were questioned about their favourite subject from Geography, Mathematics, and Chemistry. 40 liked Geography, 36 like Mathematics, 30 liked Chemistry, and 10 liked all three subjects. If 4 liked both Geography and Mathematics, and assuming that each student liked at least one subject, how many liked Chemistry only? (2 marks)

6.5.3: (NECTA 2001) Draw a Venn diagram and shade the portion corresponding to the set

$$(S \cap T') \cup (S \cap T)$$

(2 marks)

6.5.4: (NECTA 2000) By using set operations, simplify the following $(A \cup B)' \cap (A \cap B)'$. (2 marks)

6.5.5: (NECTA 2000) Out of 130 students of a certain school, 10 of them study economics and mathematics while 20 study neither of these two subjects. Those who study economics alone are three times as many as those who study mathematics only. How many students study economics? (4 marks)

6.6 Interest and Exponential Functions

Often, when you put money in a bank, or if you borrow money from a bank, there is interest. The interest (hopefully) makes up for inflation, and can make it nice for whoever is getting it.

Definition: *In a loan or bank deposit, the* principle *is the initial amount of money. The principle is the starting amount.*
The amount *is the current amount of money.*
The interest rate *is a percentage per time. Usually the annual rate is given, meaning per 1 year. In* simple interest*, the interest rate is the percentage of the principal that is added to the amount. For* compound interest*, the interest rate is the percentage of amount that is added to the amount.*

For simple interest, only the principal earns interest. Thus the amount $A = P + Pni$ where P is the principle, i is the interest as a decimal, not a percentage, and n is the number of interest time periods that have passed. This can be simplified as:

$$\text{Simple Interest :} \qquad A = P(1 + in)$$

Ex 1: *You take out a loan of* $600,000/=$ *for 2 years at* 8% *annual simple interest. After these 2 years, how much do you need to pay the bank?*

Solution: 'Annual' means 1 time per year. (Semi-annual means 2 times per year, because 'semi' means half.) Thus $P = 600,000$, and expressing the interest rate as a decimal we get $i = 0.08$, and after 2 years means $n = 2$. Therefore

$$\begin{aligned} A &= P(1 + in) \\ &= 600,000(1 + 0.08 \cdot 2) \qquad = 696,000 \end{aligned}$$

You will owe the bank 696,000/=. ✠

Simple interest pays only on the principle. It is usually used only for very short-term loans. Much more common is compound interest, where the interest already earned also earns interest.

For compound interest is often described at an annual rate, which we call i like before, but it is calculated several times per year. The following formula is good for compound interest:

$$\text{Compound Interest :} \qquad A = P\left(1 + \frac{i}{m}\right)^{mt}$$

where m is the number of times it is compounded per year, t is the number of years, i is the annual interest rate as a decimal (not percentage!), P is the principle, and A is the amount.

Ex 2: *You take out the same loan of* $600,000/=$ *for 2 years at another bank. At this one you must pay* 8% *annual interest which is compounded 4 times per year. After 2 years, how much do you owe the bank?*
Solution:

$$A = P\left(1 + \frac{i}{m}\right)^{mt} \qquad = 600,000\left(1 + \frac{0.08}{4}\right)^{4\cdot 2} = 702,996$$

You will owe the bank 702,996/=. ✠

Then, there is continuously compounded interest. In this case, the we take the limit as m approaches ∞. This introduces e to the equation, because one definition of e is

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{x}$$

We get the formula:

$$\text{Contiuously Compounded Interest :} \qquad A = Pe^{it}$$

Ex 3: *You take the same 600,000/= loan for two years, this time the interest rate is* 8% *continuously compounded. After 2 years, how much money do you owe the bank?*
Solution:

$$\begin{aligned} A &= Pe^{it} \\ &= 600,000 \cdot e^{0.08\cdot 2} \\ &= 704,107 \end{aligned}$$

You owe the bank 704,107/=. ✠

With compound interest, amounts can get very big, very fast. Albert Einstein even said ‘Compound interest is the most powerful force in the Universe.’ This is how credit card companies make money. They give you a card, and you can buy things with it, and the company pays for the things. Then, at the end of the month they send you a bill. It is very nice because you can buy so many things and you do not need to pay until the end of the month. But then, if you are late in paying, they charge you interest. Often very high interest, maybe 15% for the first month, and if you still cannot pay, maybe 30% for the next month. And it is compound interest. And very soon you owe lots of money. So a credit card is a very nice thing, if you pay your bill on time. If you are late with the bill, you will lose all your money.

Many things have exponential functions, in physics, in chemistry, in biology, and as the interest examples have already shown, in economics. An exponential function is anything that has e with an exponent. Examples in physics include the radioactive decay equation, $N =$

$N_o e^{-\lambda t}$, where N is the amount of radioactive isotopes remaining, N_o is the original amount, t is time, and λ is the decay constant. Or, for absorption of x-rays, used both in physics and medicine, $I = I_o e^{-\mu x}$, where I_o is the intensity of x-rays that hit a body, I is the intensity of the rays after passing through the body, μ is a constant that depends on the material, called the absorption coefficient, and x is the thickness of the material. In physiology, $Y = Y_o e^{-kt}$ can be the concentration of some medicine in the bloodstream at time t, where Y_o is the original concentration and k is a constant related to how efficiently the organs are cleaning the blood.

The interesting thing about exponential functions is that, if they are decreasing, there is a time called the 'half-life', which is constant, and after every half-life the amount is cut in half. (In the case of x-ray absorption it is a half-distance, a length, not a time.) If the function is increasing, then there is a doubling time, which is also constant. We will use the example of radioactive decay.

Ex 4: *Find the radioactive half-life of an isotope in terms of its decay constant λ.*
Solution: We start with the radioactive decay equation,

$$N = N_o e^{-\lambda t}$$

And then we think. After one half-life, the amount remaining N should be half of the original amount. $N = \frac{1}{2}N_o$. Thus

$$\begin{aligned}
\frac{1}{2}N_o &= N_o e^{-\lambda t} \\
\frac{1}{2} &= e^{-\lambda t} \\
\ln\frac{1}{2} &= -\lambda t && \text{Applying natural log to both sides,} \\
-\ln\frac{1}{2} &= \lambda t \\
\ln 2 &= \lambda t && \text{Because } \left(\frac{1}{2}\right)^{-1} = 2, \\
\frac{\ln 2}{\lambda} &= t
\end{aligned}$$

Using a calculator, you can find that $\ln 2 \approx 0.69315$. ✠

6.6.1 Exercises

6.6.1: You invest 300,000/ = in a bank that will pay you 10% compound interest. After 5 years, how much money will you have
(a) If the interest is compounded annually (once per year)?
(b) If the interest is compounded quarterly (4 times per year)?
(c) If the interest is compounded monthly (12 times per year)?

6.6.2: You have a credit card, and you buy a pikipiki for 800,000/ =. Unfortunately, at the end of the month, you can pay only 600,000/ =, so you still owe the credit card company 200,000/ = (this is the principle). On this debt, they charge you 15% *monthly* (not annual) interest, compounded 2 times per month. After 2 months of this, how much do you owe the credit card company?

6.6.3: A businessman is choosing investments. He has 1,000,000/ = to invest. Bank S offers him 15% annual simple interest. Bank C offers him 10% annual compound interest, compounded twice per year. Bank E offers him 9.5% annual compound interest, compounded continuously. Which bank should he choose (a) if he plans to wait 10 years? (b) if he plans to wait 30 years?

6.6.4: A certain malaria medicine has a absorption constant $k = 0.231$ days^{-1}. What is the half-life of its concentration in the blood stream?

6.6.5: A biologist isolates a bacteria and keeps it in a petri dish. She finds that every 4.6 hours, the bacteria population has doubled. Write an equation for the bacteria population P in terms of its initial population P_o.

6.6.6: (NECTA 2006) Juma wants to invest Sh. 150,000 at a rate of 10% compounded annually and accumulate the principal to Sh. 250,000. Using a calculator with a log key, find how long this will take given that:

$$S = P(1+i)^n$$

Where $i =$ interest, $n =$ number of years, and $P =$ principal. (6 marks)
(*Note:* To be technically correct, Juma's principal does not change. Rather, it is his amount that accumulates to 250,000/=.)

6.6.7: (NECTA 2002) The population of a sample is given by $P(t) = 10.000e^{0.4t}$, where t is in years. Use a non-programmable calculator to find the time to the nearest whole number on which the population of the sample will double. (6 marks)

Appendix A

The Greek Alphabet

Here are all the letters of the Greek alphabet, which we use so much as variables. You can see here the CAPITAL, lowercase, name, and common uses for each letter.

A, α	alpha	angles, angular acceleration
B, β	beta	angles, feedback
Γ, γ	gamma	surface tension, ratio of specific heats
Δ, δ or ∂	delta	changes and differences, BIG or small
E, ϵ	epsilon	very small amount, electric field constant
Z, ζ	zeta	
H, η	eta	viscosity
Θ, θ	theta	angles, temperature
I, ι	iota	
K, κ	kappa	constants
Λ, λ	lambda	wavelength, half-life
M, μ	mu	coefficient of friction, magnetic field constant, linear density
N, ν	nu	
Ξ, ξ	xi	
O, o	omicron	
Π, π	pi	3.1415926...
P, ρ or ϱ	rho	density (per volume)
Σ, σ	sigma	SUMS and density per area
T, τ	tau	torque or moment
Υ, υ	upsilon	
Φ, ϕ or φ	phi	FUNCTIONS, FLUX and angles, especially phase angles
X, χ	chi	statistical test
Ψ, ψ	psi	FUNCTIONS
Ω, ω	omega	RESISTANCE and angular frequecy or angular velocity

Appendix B

Notation

Sets

The set of all real numbers is written $\mathbb{R}$. Sometimes this is modified to $\mathbb{R}^+$ for positive real numbers or $\mathbb{R}^-$ for negative real numbers. The set of integers, $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ is written $\mathbb{Z}$. The 'null set' or 'empty set', which has no elements, is written $\varnothing$. If you want to take things out of a set, you can use a backslash, $\setminus$. For example $\mathbb{R} \setminus 0$ is the set of all real numbers except 0.

Sets are usually called capital letters, A, B, C, etc. The complement of A can be written as $A^{\complement}$ or A'. In defining sets $\in$ means 'in', and a colon, ':', means 'such that'. Thus $\{x \in \mathbb{R} : x > 4\}$ is the set of all real number x such that x is greater than 4. Another example for the use of $\in$ is $\pi \in \mathbb{R}$, pi is in the real numbers.

The set operations are $\cup$ union, and $\cap$ intersection.

General Math

Sometimes common phrases are abbreviated. 'Therefore' can be written as $\therefore$, 'implies that' is $\Rightarrow$, and 'if and only if' as $\Leftrightarrow$. Many people write a 'booyah box', $\blacksquare$, at the end of a proof to celebrate, 'Booyah! I finished the proof!'

Relations include $=$, $>$, $\geq$, $<$, $\leq$. Any relation with a slash / through it means 'not'. Thus $\neq$ is not equal to and $\notin$ means 'not in'. Approximately equal to is $\approx$ or $\simeq$. Proportional to is $\propto$. Much less than is $\ll$ and much greater than is $\gg$.

A function f is often written $f(x) = \ldots$, but can also be written $f : x \mapsto \ldots$. The domain of $f(x)$ can be written $\mathrm{dom}(f(x))$, and the range $\mathrm{ran}(f(x))$. The composition of 2 functions, f and g is written as $f(g(x))$ or $f \circ g(x)$. The inverse of $f(x)$ is written $f^{-1}(x)$. This holds also for trigonometric functions, hence if $\sin\theta = A$, it is correct that $\sin^{-1} A = \theta$. But $\sin^{-1}\theta \neq \csc\theta = \frac{1}{\sin\theta}$. Another notation for the inverse trigonometric functions is to write 'arc' as a prefix. Thus $\sin^{-1} A = \arcsin A = \theta$, $\cos^{-1} B = \arccos B = \theta$, and $\tan^{-1} C = \arctan C = \theta$.

'Right', 'perpendicular', 'orthogonal', and 'normal' all refer to a 90° or $\pi/2$ radian angle. This can be written $\perp$. Parallel can be written as $//$.

In a fraction $\frac{n}{d}$, the top is called the 'numerator' and the bottom the 'denominator'. Logarithms of base 10 and and base e are abbreviated as $\log x$ (or sometimes $\lg x$) and $\ln x$, respectively.

The sum $a_1 + a_2 + a_3 + \cdots + a_n$ can be written as

$$\sum_{i=1}^{n} a_i$$

Vectors and Matrices

Vectors are written as lower case letters in different ways. $\vec{a}, \mathbf{a}, \hat{a}, and \underline{a}$ are all vectors. Even a squiggle '$\sim$' underline is sometimes used. Usually the hat is reserved for unit vectors, $\hat{i}$ or $\hat{p}$. The magnitude of a vector is written as $|\vec{a}|$. The magnitude squared sometimes is written $|\vec{a}|^2 = ||\vec{a}||$.

Vector operations are the dot product, $\vec{a} \cdot \vec{b}$, and the cross product $\vec{a} \times \vec{b}$. Rarely, the cross product is called the 'vector product' and is written $\vec{a} \wedge \vec{b}$. The projection of $\vec{a}$ onto $\vec{b}$ is written $\text{proj}_{\vec{b}}\vec{a}$.

Matrices are usually capital letters A, B, C, etc. The determinant of a matrix A is written as $\det(A)$ or $|A|$. The transpose is A^T, and the inverse is A^{-1}.

Calculus

The limit of $f(x)$ as x approaches a is written

$$\lim_{x \to a} f(x)$$

If under the limit it is written $x \to a^+$ then the approach is from above, from the right. If it is $x \to a^-$ then the approach is from below, from the left.

A capital delta, Δ, usually means 'change'. Thus

$$m = \frac{\Delta y}{\Delta x}$$

is defining slope as change in y divided by change in x. A lower case delta, δ, is a very small change, usually used before taking a limit to make a derivative. Another way of writing the concept of the derivative would be

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Newton's Notation: The derivative of y is y'' and the derivative of $f(x)$ is $f'(x)$, these are read 'y prime' and 'f prime of x'. Second derivatives are y'' of $f''(x)$.

Leibnitz Notation is more exact because it indicates the variable that is differentiated with respect to. $\frac{d}{dx}$ is the differential operator, it tells you to take the derivative of what follows with respect to x. For example $\frac{d}{dx}(y)$ or $fracddt(s(t))$. The derivatives, once taken, can be written as $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $dsdt$. Second derivative are written as $\frac{d^2y}{dx^2}$ because there are 2 $\frac{d}{dx}$'s, but only one y.

Other notation: Sometimes, especially for derivatives with respect to time, a dot is used. Thus, if horizontal position is x, horizontal velocity would be $\dot{x}$, and horizontal acceleration $\ddot{x}$.

For integration,

$$\int_a^b f(x)\, dx$$

is the 'integral from a to b of $f(x)$ with respect to x'. $\int$ is an integral sign, a and b are the bounds or limits of integration: a is the lower bound, b is the upper bound; $f(x)$ is the integrand, and dx indicates that x is the variable of integration.

Probability and Statistics.

The factorial of n is $n!$. The number of ways or choosing and permuting r objects chosen from n total objects, with order, is ${}^nP_r$. The number of ways of choosing r objects from n total objects, without order, is ${}^nC_r$, which is also written $\binom{n}{r}$.

The probability of an event A is written $P(A)$, the probability of A given B is $P(A|B)$.

Appendix C

Exercise Hints and Solutions

Hints and Solutions to Chapter 1

Hints and Solutions to Section 1.1

1.1.8. $\frac{2x\sqrt{y}}{y-x}$

Hints and Solutions to Section 1.2

1.2.1. S
1.2.4. **(a)** $f^{-1}(x) = \ln x$. **(b)** Answers may vary. **(c)** It is undefined.

Hints and Solutions to Section 1.3

1.3.1. **(a)** $\log_b x = y$. **(b)** $\log_2 x = 4$. **(c)** $\log_2 8 = n$. **(d)** $\ln \pi = x$. **(e)** $\log a = z$. **(f)** $\log_{xy} ab = n + 1$. **(g)** $\log_2 4 = 2$. **(h)** $\log_2 1024 = 10$.
1.3.2. **(a)** $4 = 2^2$. **(b)** $c = b^8$. **(c)** $e^{1.06471} = 2.9$. **(d)** $e = 1.71828$. **(e)** $10^x = 4$. **(f)** $g^2 = ab$.
1.3.3. **(a)** 0.7231. **(b)** -0.4935. **(c)** 4. **(d)** 0. **(e)** -0.693. **(f)** 0.693. If x is less than b then $\log_b x$ is negative.
1.3.4. **(a)** $x = 3$. **(b)** $x = 2$. **(c)** $x = 4/7$. **(d)** $x = 8/9$.
1.3.6. **(a)** $\ln y = 4\ln(x^2 - 3) + 8\ln(x^3 - 1) - 12\ln(x + 1)$. **(b)** $\ln y = 4\ln(\sin x) + 3\ln(\cos x) + 2x - \frac{3}{2}\ln(x^3 - 1)$.
1.3.7. **(a)** $x = 3^{1/4}$. **(b)** $x = 10^{10^{16/3}}$ **(c)** $x = 256$
1.3.10. 4
1.3.11. Hint: make N the subject. Answer: $N = 8$
1.3.12. (a) 0.
(b) i. $2\log 2 + \log 3 = 1.0791$
ii. Hint: knowing $\log_{10} 3$ does not help, but you also know $\log_{10} 10$.
Answer: $1 - \log 2 = 0.6990$
1.3.13. $x = 10^{-1} = 0.1$
1.3.14. Hint: combine all into one logarithm, then you will find $x = 3$.
1.3.15. $y = x^3.25$
1.3.16. Hint: First let $z = \log_x 3$ and find z.
Answer: $x = 3$ or $x = 3^{-1/4}$
1.3.17. Hint: $a^{x+1} = a^x \cdot a$
Answer: $-17/9$
1.3.18. (a) 1.49752, (b) 21.4644
1.3.19. (a) $x = 9$, (b) Hint: Change everything to logs base 3, then substitute $y = \log_3 x$.
Answer: $x = 3$ or $x = 27$.

Hints and Solutions to Section 1.4

1.4.1. **(a)** $C = (5, 2)$, **(b)** $m = 3/2$

Hints and Solutions to Section 1.5

1.5.1. Hint: Multiply through to put everything in terms of $\sin\theta$, then use the quadratic equation to get 2 possible values of $\sin\theta$. Then use the unit circle and a calculator to find all 4 possibilities for θ.
Answer: $\theta = 18°$ or $162°$ or $-54°$ or $-126°$.
1.5.2. $x = 20°$
1.5.3. $\frac{x}{2} = 2y - y^2$
1.5.4. $\theta = 60°$
1.5.5. $b^2(1 + c^2) = a^2c^2$
1.5.6. $x = 90°$ or $210°$ or $270°$ or $330°$
1.5.7.

Hints and Solutions to Section 1.7

1.7.5. Hint: start with $\tan(2A + B) = 1$. Use identities to expand this expression.

Hints and Solutions to Chapter 2

Hints and Solutions to Section 2.1

2.1.1. (a) 0, (b) 1, (c) ∞, (d) $-\infty$.
2.1.2. (a) Does not exist, (c) $2x$, (d) $3x^2$.
2.1.3. (c) Does not exist.

Hints and Solutions to Section 2.2

2.2.1. $f'(x) = 2x - 1$
2.2.2. $g'(x) = 2x + 1$
2.2.3. $y' = \frac{1}{2\sqrt{x+1}}$
2.2.4. $f'(x) = 3x^2$
2.2.5. $h'(x) = 3x^2 - 2x$
2.2.6. $2x - \frac{1}{2}$
2.2.7. $3x^2 - 6x + 1$

Hints and Solutions to Section 2.3

2.3.1. **(a)** $f'(x) = 3x^2$, **(b)** $g'(x) = 6x^2 + 6x$, **(c)** $f'(x) = 4x$, **(d)** $g'(x) = 4x$.
2.3.2. **(a)** $y' = 5x^4 - \frac{1}{2}x^2$, $y'' = 20x^3 - x$; **(b)** $R'(\theta) = 10\theta - 3$, $R''(\theta) = 10$; **(c)** $f'(x) = 11x^{10} - 1$, $f'(x) = 110x^9$; **(d)** $s'(t) = 3$, $s''(t) = 0$.
2.3.4. (a) $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$, (b) $g'(x) = \frac{-1}{x^2}$, (c) $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$, (d) $g'(x) = -4x^{-5}$.
2.3.5. Hint: a horizontal tangent means the gradient is 0.
Answers: (c) $x = -2$ and $x = -6$, (d) No horizontal tangents.

Hints and Solutions to Section 2.4

2.4.1. (a) $f'(x) = \cos x + 6x$, (d) $X'(t) = -4t^{-5} - t^{-1}$.
2.4.2. Hint: for (c) use logarithm rules to simplify first.
Answers: (a) $f'(x) = \sec^2 x + \sin x$, (c) $h'(x) = 2/x$.
2.4.3. (b) $g'(x) = 6x + 4x^{-2} + x^{-1}$, (d) $F'(r) = \frac{q_1q_2}{4\pi\epsilon_o r^2}$

Hints and Solutions to Section 2.5

2.5.1. (c) $y' = (3x + 4)\cos x + 3\sin x$
2.5.2. (c) $f'(x) = (4x - 1)(8x^3 - x^2 + 1) + (24x^2 - 2x)(2x^2 - x)$
2.5.3. (a) $g'(x) = e^x \sin x + e^x \cos x$, (b) $y' = \cos^x - \sin^2 x = \cos(2x)$, (c) $R' = \cos\theta$, (d) $y' = -15x^{-4}\cos x - 5x^{-3}\sin x$.
2.5.4. (a) $y' = 6 + 6\ln x$, (b) $f'(x) = 2(x^3 + 3x - 1)(3x^2 + 3)$, (c) $g'(x) = 2x$, (d) $h'(x) = (\frac{1}{3}x^3 - 2x)\sec^2 x + (x^2 - 2)\tan x$.
2.5.5. (a) $y' = 2 + 2\ln x$
(b) Hint: $a^{x+y} = a^x \cdot a^y$
(c) $y' = 2\cos(2\theta)$
2.5.6. (c) $y' = \frac{(3x+1)4x-6x^2}{(3x+1)^2}$
2.5.7. (c) $y' = -\csc^2\theta$
2.5.8. (a) $f'(x) = \frac{15x^2(x+1)-(5x^3-2)}{(x+1)^2}$, (b) $g'(x) = \frac{1-\ln x}{x^4}$, (c) $y' = 2x$, (d) $H' = \frac{4ye^y-2y^2e^y}{e^{2y}}$.
2.5.9. (a) $Q' = \frac{(x^2-1)\frac{1}{2}x^{-1/2}-2x(1+\sqrt{x})}{(x^2-1)^2}$, (b) $y' = 0$.
2.5.12. (a) $y' = \big[(2x-5)[(x^2+1)(6x^2-2x) + 4x^4 - 2x^3] - 2(x^2+1)(2x^3 - x^2)\big] \cdot (2x-5)^{-2}$,
(b) $y' = e^x \cdot \frac{(2x^2-1)(5x+5)-20x^2}{(2x^2-1)^2}$,
(c) $y' = \frac{3x}{\cos x} + \frac{6x\cos x + 3x^2\sin x}{\cos^2 x} \cdot \ln x$.
2.5.13. First, find $f'(x) = \frac{4x^2+1}{2x^2}$, then $f'(1) = 5/2$ is the gradient.

Hints and Solutions to Section 2.6

2.6.1. (a) $y' = -12(4-2x)^5$, (b) $f'(x) = -6(x+1)\sin(3x^2+x)$, (c) $g'(x) = 4\tan^3(x)\cdot\sec^2(x)$, (d) $y' = \frac{1}{2}(3x^2-1)(x^3-x+1)^{-1/2}$.
2.6.2. (a) $v' = -12\sin(3t-6)$, (b) $h' = \frac{8x-1}{4x^2-x}$, (c) $y' = 3e^{3x+1}$, (d) $X' = 8t(2+t^2)^3$.
2.6.3. (a) $y' = (27x^2 - 18x + 9)(x^3 - x^2 + x)^8$, (b) $y' = -(24t-4)(3t^2-t)^{-5}$, (c) $y' = 2x\cos(x^2)$, (d) $y' = 2\sin x\cos x$.
2.6.4. (a) $y' = 3x^2\cos(x^3)$, (b) $y' = 3\sin^2 x \cdot cosx$, (c) $y' = 4x^3\cos(x^4)$, (d) $y' = 4\sin^3 x \cdot cosx$.
2.6.5. (a) $f'(x) = x^2(x^3+1)^{-2/3}$, (b) $g'(r) = \frac{3r+2}{r^2+r}$, (c) $h'(t) = 12te^{2t^2}$, (d) $y' = -2x\sin(x^2)$.
2.6.6. (a) $y' = -(2x+1)\cos(x^2+x)$, (b) $f' = -\sin(\sin(x))\cdot\cos(x)$, (c) $g' = -4x\cos(x^2)\cdot\sin(x^2)$, (d) $A'(t) = iPe^{it}$.
2.6.7. (a) $y' = xe^x + e^x$, (b) $f'(x) = x^2e^x + 2xe^x$, (c) $g'(x) = e^{2x} + 2xe^{2x}$, (d) $h'(x) = 2x^2e^{2x} + 2xe^{2x}$.
2.6.8. (a) $y' = 2e^{2x}(x)(x+1)$, (b) $f'(x) = \frac{2\sqrt{x^2+1} - 2x^2(x^2+1)^{-1/2}}{x+1}$, (c) $g'(x) = \cos^2 x - \sin^2 x$, (d) $h'(x) = -\cos(\cos x)\cdot\sin x$.
2.6.9. (a) $y' = \frac{2\cos(x^2)\cos(2x)+2x\sin(2x)\sin(x^2)}{\cos^2(x^2)}$, (b) $f' = \frac{-4(x^2-1)\cos^3(x)\sin(x)-2x\cos^4(x)}{(x^2-1)^2}$, (c) $g' = 2\sin(x)\cos(x)[\cos^2(x) - \sin^2(x)]$, (d) $h' = 2\cos(2x)\tan x + \sec^2 x\sin(2x)$.
2.6.10. (a) $y' = \frac{6x}{3x^2-1} - \frac{1}{x} = \frac{3x^2-1}{3x^3-x}$, (b) $y' = \frac{4}{x+1} - \tan x = \frac{4\cos x-(x+1)\sin x}{(x+1)\cos x}$.
2.6.11. $y' = -3\cos^2 x\sin x$
2.6.12. Hint: First use logarithm rules to simplify.
Answer: $y' = \frac{3}{3x-2} - \frac{1}{x+1}$
2.6.13. $(1+3x)e^{3x}$
2.6.14. $\frac{2\cos(2x)}{\sin(2x)}$
2.6.15. $\frac{4}{3}x^{-2/3}\csc(\sqrt[3]{x})$

Hints and Solutions to Section 2.7

2.7.2. Hint: for (c) and (d) expand first. Or use the Chain Rule.

Hints and Solutions to Chapter 3

Hints and Solutions to Section 3.1

3.1.3. $y = x^2 - x$

Hints and Solutions to Section 3.2

3.2.1. $\int \sec^2 x = \tan x$, but $\int \sec x$ is undefined.
3.2.2. $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}$

Hints and Solutions to Section 3.3

3.3.1. $\frac{512}{81}$
3.3.2. $1 - \frac{\cos x}{a}$

Hints and Solutions to Section 3.4

3.4.7. $8 - \frac{3}{e} + 2e$
3.4.10. $-\frac{1}{3}\cos^3 x$

Hints and Solutions to Section 3.8

3.8.1. $\frac{1}{36}\left[(2b^3+3)^6 - (2a^3+3)^6\right]$

Hints and Solutions to Chapter 4

Hints and Solutions to Section 4.1

4.1.1. (a) $-3\hat{\imath} + 4\hat{\jmath}$
(b) 5
(c) $\frac{-3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$
4.1.2. $m = 5,\ n = 1$
4.1.3.
4.1.4. (7/3, 7/3)

Hints and Solutions to Section 4.2

4.2.4. If they are perpendicular, then $(\vec{a} - \vec{b}) \cdot (\vec{a} + \vec{b}) = 0$

Hints and Solutions to Section 4.4

4.4.1. Hint: Use the coordinates to write position vectors for the Hydrogens and the Carbon, then subtract position vectors to get displacement vectors, for example $\vec{H}_1 - \vec{C}$ might be the displacement vector from Carbon to one Hydrogen. Then dot these displacement vectors and solve for the angle between them.
4.4.3. Hint: Remember, $\vec{F} = m\vec{a}$, and 'unit mass' means that mass $m = 1$.
4.4.4. The position vector of $\vec{Q}$ relative to $\vec{P}$ is given by $\vec{Q} - \vec{P}$. And you know how to find velocity and acceleration from position... differentiate!

Hints and Solutions to Section 4.5

4.5.1. (c) Undefined.
4.5.2.
4.5.3.
4.5.4.
4.5.5. (a) Undefined,
4.5.7.
4.5.9. $x = -4$, $y = 11$
4.5.10. $AB = I =$
1 0 0
0 1 0
0 0 1

Hints and Solutions to Section 4.6

4.6.2. $T(0, 0) = (0, 0)$.
4.6.4. $k = 4$

Hints and Solutions to Section 4.8

4.8.2. The positive x-axis in vector form is in the direction $\hat{\imath} + 0\hat{\jmath} + 0vk$.

Hints and Solutions to Chapter 5

Hints and Solutions to Section 5.1

5.1.1. $5/36 = 0.139$
5.1.2. i. $1/27 = 0.037$, ii. $4/27 = 0.148$.
5.1.3. 0.765
5.1.4. 3326400
5.1.5. Hint: 'At most 1 which is defective' means that either 9 out of 10 are not defective OR all 10 are not defective. If 9 of 10 are not defective, be sure to CHOOSE which one is defective. Answer: 0.736
5.1.6. 60
5.1.7. (a) They are not mutually exclusive. (b) They are mutually exclusive.
5.1.8. 0.6
5.1.9. 0.4196
5.1.10. (a) 0.6, (b) 0.3
5.1.11. $P(A) + P(B) = 0.91 \neq 0.78$ thus the events are not independent.
$P(A) + P(B) \neq 1$ thus the events are not mutually exclusive.
5.1.12. 0.222

Hints and Solutions to Section 5.4

5.4.1. (a) 0.322, (b) 0.0436, (c) 0.0873.

Hints and Solutions to Section 5.5

5.5.5. $\bar{x} = 89.65\text{kg}$, $\sigma =$

Hints and Solutions to Chapter 6

Hints and Solutions to Section 6.4

6.4.2. (a) $y^6 + 6y^5x + 15y^4x^2 + 20y^3x^3 + 15y^2x^4 + 6yx^5 + x^6$
(c) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

Appendix D

Brief Health Matters

Some Facts about HIV/AIDS

How to Properly Use a Condom

To be effective, you must use a condom every time you have sex. A condom is good for one use, do not use the same condom more than one time.

A condom should be used anytime semen or vaginal fluids are exchanged.

To use a condom properly, first make sure the package is not torn and that the date of expiration is not passed.

When the penis is erect, open the packet carefully. Do not use a sharp object (teeth, razor blade, knife, scissors) to open the packet because you might tear the condom.

Pinch the tip of the condom so that air is not trapped inside, then unroll the condom down over the penis.

After having sex, make sure that the condom is still on the penis when the penis is removed from the vagina. Immediately remove the condom, being careful not to spill the sperm inside. Tie a knot near the base of the condom to close it. Throw it down a pit-toilet or into a burn pile. (It is best not to flush the condom down a flush toilet as it may become stuck in the pipes.)

This is so important that I will write the entire thing is Kiswahili, too.

Wakati wa kutumia kondomu ya kime ni muhimu kufuata hatua zifuatazo. Hakikisha kwamba pakiti inayoihifadhi haiapasuka na kwamba tarehe ya kuisha muda wake haijafika.

Kama uume umedinda, fungua pakiti kwa uangalifu. Usitumie wembe, meno, mkasi, au kisu! Minya sehemu ya juu kutuoa hewa ndani ya kondomu wakati wa kuvalisha uume, ili kuzuia kupasuka kwa kondomu wakati wa kujamiiana. Visha uume kondomu taratibu mpaka uufunike wote. Ukiwa na uhakika kwamba kondomu imevishwa inavyotakiwa, unaweza kukutana kimwili na mwanamke.

Wakati wa kutoa uume kutoka ukeni, uwe mwangalifu kwamba kondomu bado ipo sehemu inayopaswa kuwa. Baada ya kutoa uume kutoka ukeni, vua kondomu kwa uangalifu uume ukiwa bado umedinda, ili kuepuka shahawa zisimwagikie ukeni. Tupa kondomu iliyotumika kwenye choo cha shimo au uichome. Usitupe kondomu iliyotumika ovyo.

Either the man or the woman can put the condom on the penis. Both can put it on, both can buy it.

Mwanamume au mwanamke anaweza kuweka kondomu uumeni. Wote wawili wanaweza kuiweka, wote wawili wanaweza kuinunua dukani.

Condoms must be used only once, because if they are used twice or three times they will not protect well from pregnancy and sexually transmitted diseases. Condoms must not be washed and reused. They must be thrown into a pit-toilet or burnt after 1 use. Condoms are made to be used only once.

Kondomu inapaswa kutumika mara moja tu kwa kila mshinda wakati wa kujamiiana, kwa sababu ikitumika mara 2 au 3, uhakika wa kuzuia mimba wala uambukizo wa magonjwa ya zinaa unatoweka. Kondomu isifuliwe ali mara baada ya kutumika itupwe kwenye choo cha shimo au ichomwe moto. Kondomu zimetengenezwa kwa kutumiwa mara 1 tu.